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A Move to More Fruitful Ground? Analyzing the Challenge of Compromise Platonism in the Philosophy of Mathematics

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W. V. QUINE, HILARY PUTNAM, AND KURT GÖDEL present arguments for mathematical realism that have come to dominate Platonist philosophies of mathematics. More recently, Penelope Maddy has argued that mathematics as understood in Quine/Putnam Platonism and Gödel Platonism stands in contrast to actual mathematical practice. In *Realism in Mathematics*, Maddy seeks to reconcile this tension— presenting a "compromise Platonism" that purports to incorporate the best of both Quine/Putnam Platonism and Gödel Platonism. To this end, she provides a compelling naturalistic account of mathematical intuition and extrinsic justification that accurately reflects the practice of mathematics. Most important, because Maddy allows for such varied extrinsic and intrinsic forms of mathematical support, she leaves as a challenge the task of determining exactly how we should evaluate conflicting forms of justification.

However, in my opinion Maddy's compromise Platonism leaves us ill equipped to face this challenge. More specifically, it is difficult to see how, when intrinsic and extrinsic supports disagree, we can proceed without falling subject to the same criticisms that motivated compromise Platonism in the first place. We begin with a discussion of Quine/Putnam Platonism and Gödel Platonism.

Quine/Putnam Platonism argues that mathematics and the physical sciences are inextricably linked. Truly, the physical sciences are littered with mathematical references. From the Law of Universal Gravitation to the structure of DNA, the physical sciences use mathematical language to describe physical phenomena. But could we form a physical theory of the world without the use of mathematics? Quine and Putnam argue no:

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the physical sciences' use of mathematics is more than superficial; rather, mathematics provides the framework that makes possible description and prediction of physical phenomena that wouldn't be possible otherwise.

As such, Quine/Putnam Platonism takes knowledge of mathematical entities to be justified by the role they play in the physical sciences not by self-evidence or intuition. For Quine/Putnam Platonism, mathematical objects must exist because they are indispensable for our best theory of the world. It is this ability, then, of an area of mathematics to provide a successful framework for the physical sciences that justifies our belief in this area of mathematics' truth. Putnam writes:

> A mathematical theory that has become the basis of a successful and powerful scientific system, including many important empirical applications, is not being accepted merely because it is 'intuitive', and if someone objects to it we have the right to say 'propose something better!' ("Mathematics" 303)

Thus, for Quine/Putnam Platonism, the *application* of mathematics to physical science is its only justification.

The Quine/Putnam version of Platonism draws three notable objections, the first of which observes that under the Quine/Putnam view, mathematics loses its a priori and non-empirical status. As discussed above, Quine/Putnam Platonism sees mathematics as justified by its role in our physical theory of the world. However, as any empirical scientist will tell you, these physical theories are, at best, working hypotheses, not a priori truths. Thus, if mathematics is justified by its usefulness in a hypothesis that is subject to change, then the basic mathematical "truths" we take as a priori are also subject to change. In all, Quine/Putnam Platonism is, at once, troublesome for mathematicians who believe they are discovering necessary truths about the world. Indeed, if Quine/Putnam Platonism is correct then, like the empirical theories, a mathematical theory is "essentially a working hypothesis in one form or another…the sands are constantly shifting, one never finds absolutely solid ground on which to stand" (Geroch 92).

Second, Quine/Putnam Platonism fails to account for areas of mathematics that seem obvious, immediately self-evident. In the empirical sciences, the use of even the most elementary mathematics comes at a high theoretical level. The most basic physical statements make little or no use of mathematics. At higher theoretical levels, physical statements gain more mathematical content. However, if even the most elementary mathematics is not "useful" for basic physical statements, then facts like 2+5=7 should become apparent to us only when we begin to investigate these higher theoretical physical statements. But as any schoolchild will tell you, that 2+5=7 was *obvious* way before s/he ever began making highly theoretical physical statements. Thus, Quine/Putnam Platonism fails to harmonize with the obviousness of elementary mathematics.

Third, under Quine/Putnam Platonism, unapplied mathematics is entirely unjustified. In the actual practice of mathematics, mathematicians do not develop mathematics only for its direct application to empirical science (though many certainly do). Unapplied mathematics is just as "true" as applied mathematics. Thus, in the practice of mathematics, the application of mathematical theory to empirical science is certainly stroke of good fortune, but never an absolute necessity. Quine/Putnam Platonism fails to respect this reality of the practice of mathematics.

Gödel Platonism takes a substantially different stance, drawing an analogy between mathematics and the empirical sciences. Gödel posits the faculty of mathematical intuition as a "psychological fact...sufficiently clear to produce the axioms of set theory" (484). This intuition, then, allows us to discover those basic mathematical truths (from set theory in particular) that are self-evident. For Gödel, this intuition for mathematics is analogous to perception for physical science. Indeed, Gödel does not "see any reason why we should have less confidence in this kind of perception, i.e., in mathematical intuition, than in sense perception...[for] physical theories" (484). In all, exactly the obviousness of elementary mathematics that was so troublesome under Quine/Putnam Platonism is justified under Gödel Platonism by this faculty of intuition.

Extending this analogy, Gödel also argues that some mathematical postulates, if not intuitable, can still be justified on account of the postulate's (1) intuitively evident consequences, (2) usefulness in the physical sciences, or (3) fruitfulness within mathematics. For Gödel, the truths established by the intuition are not the only mathematical axioms we are justified in accepting. Axioms in the first category (1) are justified because they explain or systematize our intuitive data. Just like our model of electrons, protons, and neutrons systematizes and explains physical phenomena observable using perception, an axiom is justified when it systematizes or explains some mathematically intuitive truth. Quine/Putnam-style indispensability justifies axioms falling under (2). Perhaps most interesting, Gödel insists that an axiom may be justified (3) if it yields powerful and elegant tools within mathematics. In particular, Gödel argues that a new axiom is justified if it has "consequences demonstrable without the new axiom, whose proofs with the help of the new axiom, however, are considerably simpler and easier to discover" (477). In all, Gödel's system has bipartite supports: an area of mathematics is justified by either its self-evidence to the intuition or its fruitfulness in mathematical or empirical science.

Like the Quine/Putnam Platonism before it, Gödel's Platonism is subject to some common objections. First, Gödel Platonism fails to give an account of exactly how this "mathematical intuition" operates. Even more basic, Gödel needs to give a cogent argument for this intuition's existence and uniformity in humans. Second, the variety of methods for justifying an axiom deserve a more careful analysis. For example, should a new axiom really be justified because of its aesthetic value in simplifying a proof? After all, Gödel provides no argument demonstrating that mathematical objects like sets are more likely to have tidy elegant properties. In all, both Quine/Putnam Platonism and Gödel Platonism are open to some cogent criticism.

Maddy's own contribution, compromise Platonism, attempts to avoid such criticisms by taking parts from both Quine/Putnam Platonism and Gödel Platonism. To this end, compromise Platonism argues that we do have a faculty of mathematical intuition along similar lines to what Gödel proposed; however, this intuition is formed through the cognitive process of visual perception of sets in the physical world. In allowing intrinsic support, Maddy hopes to avoid Quine/Putnam Platonism's unfaithfulness to mathematical practice. At the same time, Maddy adopts the Quine/Putnam indispensability arguments as an alternative form of justification for mathematical postulates. Justification from intuition Maddy calls intrinsic. Justification from indispensability Maddy calls extrinsic. In the end, Maddy argues that extrinsic justification gives us reason to believe mathematics is a science on par with natural sciences. Intrinsic justification, though, accounts for the unique practice of the mathematics. Let's take a moment to spell out compromise Platonism in more detail.

Maddy begins with a naturalized account of intuition in perception. Cognitive scientists note that the ability to recognize a triangle, say, is not inborn. Rather, during the first few days of sight for newly sighted humans, subjects need to carefully count edges. However, after some time subjects learn to perceive that the figure is a triangle quickly. Obviously, something different happens when the subject originally looks at the triangle and needs to count sides and when the subject identifies the triangle immediately. Maddy cites work by D. O. Hebb (Hebb 122–24) in showing this learned ability to perceive the triangle immediately is characterized by sets of nerve cells becoming more efficacious to their neighbors firing. Thus, without getting into the details, we can say that in the process of learning to immediately recognize a triangle, the subject gains a measure of intuition about triangles in general. This intuition is, physically, the pattern of cells that fire when encountering triangle-like objects in the field of vision—nerve cells that have become efficacious to one another.

By asserting that we perceive sets, Maddy argues, certain properties of sets (and their corresponding axioms in set theory) become cognitively intuitive. Under the traditional Platonist view, mathematical objects like sets are non-spatial and non-temporal. Thus, they cannot be directly perceived. Maddy argues that we perceive sets when we look into an egg carton, say, and see twelve eggs. Sets are all around us and we directly perceive them. More, in the process of perceiving sets we form a sort of intuition about sets. We notice, or rather our nerve-cells configure themselves such that we notice, that sets can be combined into new sets. We notice that sets with the same members have the same size. In short, Maddy argues that we perceive sets directly, and in the process form an intuition about the properties of sets. Further, this intuition forms the intrinsic support for some of the axioms of set-theory.

At the same time, compromise Platonism observes intuition does not fully support all of the set-theoretical axioms; rather, extrinsic justifications play an important role. Maddy cites the Axiom of Choice as a set-theoretical axiom with extrinsic support. Zermelo's Axiom of Choice does not immediately come across as intuitive. However, the axiom has fruitful consequences. Firstly, the Axiom of Choice is necessary to prove Cantor's well-ordering hypothesis. Second, without the Axiom of Choice entire areas of mathematics are unjustified. Indeed, in diverse fields of mathematics the Axiom of Choice was tacitly assumed; thus, to deny the Axiom of Choice would be to disqualify large portions of classical mathematics. In allowing extrinsic support, compromise Platonism agrees with Zermelo in noting "unprovability…is in no way equivalent to nonvalidity" (Zermelo 187).

For compromise Platonism, these extrinsic supports come in many forms. Maddy lists seven types of extrinsic support: verifiable consequences, powerful new methods for solving open problems, simplifying and systemizing theory, implying previous conjectures, implying intuitive or aesthetic truths, and strong intertheoretic connections. In identifying these seven types of extrinsic support, compromise Platonism recognizes that extrinsic support comes in a wider variety than supposed by either Quine/Putnam Platonism or Gödel Platonism.

At this point we have seen how Maddy's compromise Platonism characterizes both intrinsic and extrinsic support. Further we have seen how accepting both intrinsic and extrinsic support seems to avoid some of the objections that plagued Quine/Putnam Platonism and Gödel Platonism. Maddy even defines another type of support, rules of thumb, which cannot be intrinsic because they are not based in direct perception but are nonetheless an "intuitive feeling." How, then, do we weigh all these different types of justification when they are in disagreement?

This is the challenge Maddy proposes for compromise Platonism—to provide a way to weigh different types of support for theoretical axioms in set theory and mathematics as a whole. Maddy remarks, we are left with "a fundamental problem, the problem of describing, explaining, and evaluating non-demonstrative arguments in mathematics" (148). We can precisely formulate this challenge: given a prospective axiom, describe how we can determine the truth-value for this axiom given its differing levels of intrinsic and extrinsic support. In framing this challenge, Maddy hopes to have shifted the Platonist dialogue to more fruitful ground.

In my opinion, compromise Platonism necessarily leaves us ill-equipped to meet this challenge. More precisely, I believe that under certain circumstances, it appears highly unlikely that we can meet this challenge without falling subject to the same criticisms of Quine/Putnam and Gödel Platonism that inspired compromise Platonism in the first place. To demonstrate my concern, consider two "prospective" mathematical axioms, one from set theory (the Axiom of Choice) and one from geometry (the parallel postulate).

As discussed above, the Axiom of Choice has strong extrinsic support. The axiom is necessary for parts of important areas of mathematics like analysis. For example, without the Axiom of Choice we cannot form the Least Upper Bound axiom for real numbers. What's more, the axiom is "in principle *necessary for science*" (Zermelo 187). Clearly, the Axiom of Choice is strongly supported by several different forms of extrinsic evidence.

However, because the Axiom of Choice leads to extremely counter-intuitive results like the Banach-Tarski paradox, the *negation* of the Axiom of Choice has strong intrinsic support. The Banach-Tarski theorem shows that if we accept the Axiom of Choice, then all spheres are congruent by finite decomposition. Most counter-intuitively, the unit ball, A, is equivalent by finite decomposition to the union of two copies of itself. Obviously, such a result clashes with our common-sense intuition concerning three-dimensional volumes. Namely, the amount of water I can fit in one ball cannot fill two balls. Even Hilary Putnam saw the Banach-Tarski theorem as a plausible justification for rejecting the Axiom of Choice ("Models" 429). In all, the Axiom of Choice is clearly well supported extrinsically and well refuted intrinsically.

Our second example, the parallel postulate, has strong intrinsic support. To be sure, the parallel postulate is not, when simply stated aloud, immediately intuitive. However, this venerable postulate is necessary to form the geometric theorems that mirror common-sense geometric intuition. Indeed, without the parallel postulate triangles can consist of three right angles. Given our substantial intuition concerning geometric shapes, this seems (intuitively) obviously impossible. Without going into many examples, I believe it is fairly easy to convince oneself that the parallel postulate does have substantial intrinsic support.

However, the parallel postulate is outright refuted by extrinsic evidence. Modern physical observation of light under the influence of gravity has shown that the actual structure of our universe is non-Euclidean. Indeed, our universe if filled with warps, bumps, and holes. As such, the parallel postulate is certainly not true of the physical universe in which we live. Putnam points to the parallel postulate as an example showing that "our notions of what is 'self-evident' have to be subject to revision... in the light of new [physical] theories" ("Mathematics" 302). In all, while the intrinsic evidence for the parallel postulate is strong, the extrinsic evidence outright denies the parallel postulate.

Before us we have two cases where the intrinsic and extrinsic evidence tend to disagree. Maddy's challenge commands us to make a decision concerning these two axioms. We have three options: (1) side with the intrinsic evidence, (2) side with extrinsic evidence, or (3) decide that the axiom/postulate is neither true nor false.

The first option, siding with the intrinsic view, leaves us open to the same criticism of Quine/Putnam Platonism that we purported to avoid in compromise Platonism. If we honor the intrinsic evidence regarding the Axiom of Choice, we must reject the Axiom of Choice. Unfortunately, this implies that entire areas of mathematics (those areas that require the Axiom of Choice) are unjustified. However, we criticized Quine/Putnam Platonism for precisely the same reason—that it left large areas of (unapplied) mathematics unsupported. Thus, if we side with intrinsic evidence we make ourselves vulnerable to the same criticisms that motivated our use of intrinsic evidence in the first place. We run into a similar problem with the parallel postulate. Honoring the intrinsic evidence means affirming the parallel postulate and rejecting non-Euclidean geometry. However, this seems to imply that all the work done in non-Euclidean geometry is not justified and not mathematics. Once again we are subject to the same criticism that motivates our use of the intrinsic view in the first place.

The second option, siding with the extrinsic view, suggests we are open to the same criticisms that motivated our use of extrinsic evidence in the first place. Honoring the extrinsic evidence requires that we deny the parallel postulate and justify only non-Euclidean geometry. In rejecting Euclidean geometry, though, we reject one of the most well-studied areas of classical mathematics. If non-Euclidean geometry is true, then Euclid was not studying mathematics. Perhaps even more disconcerting, in abandoning Euclidean geometry it seems that we are not accurately reflecting the practice of mathematics. When mathematicians discover that a new area of mathematics is more physically useful, we do not simply abandon the old domain of mathematics as "wrong." Choosing option two, commits us to just such a view. Similarly, in honoring extrinsic evidence we affirm the Axiom of Choice. However, extrinsic evidence is, presumably, justified by its ability to approximate the truth about mathematical entities. However, as Maddy's naturalism dictates, mathematical entities are "present" in the structure of the world we perceive. But the world we perceive does not include balls that behave as the Banach-Tarski paradox would suggest. In short, if our intuition and extrinsic support approximate the truth of the same mathematical objects, then how can we account for their divergence on this matter? We criticized Gödel Platonism for not providing an account of exactly how the different categories of extrinsic support work. In affirming the Axiom of Choice, we make ourselves vulnerable to the same criticism.

What seems like the best option, assigning neither true nor false to the axiom/postulate, runs counter to the fundamental realist view. Given that both non-Euclidean and Euclidean geometry seem to have a place in our mathematics and that the Axiom of Choice has counter-intuitive consequences, we might consider not assigning any truth value to these axioms. Perhaps the question of whether the Axiom of Choice is true is like asking whether my Honda is red or blue when I don't even own a car. Indeed, it would be convenient to say that in situations like these, the mathematician is justified in working in a model with the Axiom of Choice or without the Axiom of Choice—likewise for the geometer and the parallel postulate.

Nevertheless, Maddy and her predecessor Gödel contend that this is not an option. Gödel observes that "Cantor's conjecture," speaking about the Continuum Hypothesis but the comment applies for all postulates, "must either be true or false" (476). Maddy likens taking the third option to the "physicist [solving] the question of the free quark by adopting a philosophy of physics according to which it is not longer a problem" (129) which is simply contrary to the spirit of science. This, at once, seems hypocritical. Maddy takes as almost a Golden Rule, that "the philosopher's job is to give an account of mathematics as it is practiced, not to recommend sweeping reform" (23). Nevertheless, the practice of mathematics clearly indicates that mathematicians see Euclidean and non-Euclidean geometry as equally true, or at the very least equally mathematical. In rejecting this third option, as most realist positions including Maddy's require, its seems that we fail to describe a fundamental aspect of mathematical practice—and describing and justifying mathematical practice is precisely Maddy's goal.

Thus, it seems highly likely that for certain types of possible axioms, we are necessarily ill equipped to meet to the challenge of compromise Platonism. In at least two cases where the intrinsic support disagrees with the extrinsic support (parallel postulate and Axiom of Choice), there is no available option for successfully meeting compromise Platonism's challenge. Indeed, based on these two examples, it seems highly likely that other possible axioms may encounter the same difficulties. Thus, with respect to at least these two axioms, it appears that when I take Maddy's challenge, I inevitably make myself vulnerable to the traditional criticisms of Quine/Putnam Platonism and Gödel Platonism that compromise Platonism sought to avoid.

In closing, certainly compromise Platonism identifies the parts of Quine/Putnam Platonism and Gödel Platonism that resonate to a large extent with the actual practice of mathematics. Further, compromise

Platonism provides a much more convincing account of mathematical intuition than was ever presented in Godel's writing However, we should not jump to the conclusion that compromise Platonism picks apart Godel and Quine/Putnam Platonism in precisely the way necessary to avoid traditional criticisms Rather, it seems that compromise Platonism may simply avoid these criticisms in the short run, but ultimately become vulnerable in meeting Maddy's challenge

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