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AN INTUITIVE AXIOMATIC DEVELOPMENT OF TRUTH-FUNCTIONAL LOGIC*

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INTRODUCTION

In an introductory logic course, little attention is given to the axiomatic development of truth-functional logic. The student is first taught the concept of validity: "in a valid argument, it is not possible for the conclusion to be false when all the premises are true" (McKay 9). With this definition in mind, the student is then taught to construct proofs to show that arguments are valid. She is given a set of rules to manipulate formulas and an explanation of how to construct proofs. How does the student know that she has enough rules to construct proofs for any valid argument? And how does the student know that she can construct proofs only for valid arguments?

Not until an intermediate logic class does the student encounter an answer to these questions. Most authors axiomatically develop a truth-functional logic, setting up a system with a minimum number of axioms and one rule of inference, modus ponens. For example, Irving Copi uses the following three axioms and one rule of inference: Axiom 1) $p \supset (p \cdot p)$, Axiom 2) $(p \cdot q) \supset p$, Axiom 3) $(p \supset q) \supset [\sim(q \cdot r) \supset \sim(r \cdot p)]$, and Rule 1) From p and $p \supset q$ to infer q (Copi 227-28). While the first two axioms and the rule of inference may be intuitive enough, the third axiom has little intuitive meaning for the beginning logic student. Furthermore, it is not intuitively obvious that these axioms allow the student to use the rules they have been using to construct proofs. Church's system, Götlind-Rasiowa's system, Frege's system, and Łukasiewicz's system all have similar unintuitive axioms (Copi 237).

In most intermediate logic textbooks, authors prove that their systems are complete and consistent. Then they show that the rules the students have been using all along to construct proofs can be derived as theorems from their axiomatic system.

In this paper, we develop an axiomatic system of truth-functional logic which more closely mirrors the actual way students are taught to construct proofs. The axioms are elementary valid arguments which allow for the introduction and elimination of the truth-functional operators. The rules of inference are structural rules which allow for the combination of these elementary valid arguments to prove other arguments are valid. The constructed system conforms to the intuition of a beginning logic student doing proofs.

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We then prove that this system is sound, compact, complete, and effectively enumerable. A system is sound (consistent) if only valid arguments have proofs. A system is compact if whenever a set of assumptions imply a result, a finite set of these assumptions imply that result; if we can construct a proof, we can trim down the proof to a finite set of assumptions. A system is complete if each valid argument has a proof. A system is effectively enumerable if there is an effective procedure for generating all the proofs of valid arguments.

DEFINITIONS

DEFINITION 1: RULES OF SYNTAX

We use the standard symbols for the truth-functional operators of *and*, *not*, *if ... then ...* , and *or*. A formula is defined to be any finite concatenation of undefined symbols in the system. However, only some of these sequences will be regarded as well-formed formulas. Well-formed formulas are the smallest set that includes the set of atomic formulas and is closed under the following rules:

1) If α is a formula then so is $(\sim\alpha)$
2) If α and β are formulas then so are $(\alpha\wedge\beta)$, $(\alpha\vee\beta)$, and $(\alpha\supset\beta)$

As a result of this definition, to show that something is true of all well-formed formulas, we simply need to show that 1) it is true of all atomic formulas and 2) if it is true of two formulas α and β, then it is true of any combination which can be formed with the above rules. This is called induction over the formulas and is similar to the way induction over the numbers works. In mathematics, to prove a property is true for all nonnegative integers, we show that: 1) the property is true for the number zero; and, 2) if the property is true for an arbitrary number, then it is true for the successor of that number.

DEFINITION 2: TRUTH ASSIGNMENT

A truth assignment is a compact way of describing a truth table. A truth assignment, v, assigns T for true or F for false to each atomic formula. Its extension, $\boldsymbol{v}$, assigns T or F to each well-formed formula as follows:

1) $\boldsymbol{v}(\sim\alpha) = \mathrm{T}$ iff $\boldsymbol{v}(\alpha) = \mathrm{F}$
2) $\boldsymbol{v}(\alpha\wedge\beta) = \mathrm{T}$ iff $\boldsymbol{v}(\alpha) = \mathrm{T}$ and $\boldsymbol{v}(\beta) = \mathrm{T}$
3) $\boldsymbol{v}(\alpha\mathrm{v}\beta) = \mathrm{T}$ iff $\boldsymbol{v}(\alpha) = \mathrm{T}$ or $\boldsymbol{v}(\beta) = \mathrm{T}$
4) $\boldsymbol{v}(\alpha\supset\beta) = \mathrm{T}$ iff $\boldsymbol{v}(\alpha) = \mathrm{F}$ or $\boldsymbol{v}(\beta) = \mathrm{T}$

DEFINITION 3: SATISFACTION

We say a formula is satisfied by $\boldsymbol{v}$ iff $\boldsymbol{v}(\alpha) = \mathrm{T}$. We say a formula is satisfiable iff some truth assignment satisfies it. We also say a set of formulas is satisfiable iff some truth assignment satisfies all formulas in the set.

DEFINITION 4: SEMANTICAL IMPLICATION

Now we characterize valid arguments. Let Γ be a set of formulas and let α be a formula. Then we say Γ semantically implies α, written as $\Gamma \vDash \alpha$, iff every truth assignment that satisfies all the formulas in Γ satisfies α.

DEFINITION 5: SYNTACTICAL IMPLICATION

Now we characterize proveable arguments. We say Γ syntactically implies α, written as $\Gamma \vdash \alpha$, when $\vdash$ is the smallest relation satisfying the following axioms and closed under the following rules of inference (Γ,q is defined as $\Gamma \cup \{q\}$. The symbol $\rightarrow\!\leftarrow$ is defined as a contradiction of the form $p \wedge {\sim}p$):

Axioms. The axioms can be divided into two broad classes: elimination and introduction of the operators $\wedge, \vee, \sim, \supset$. As the classifications suggest, the elimination axioms eliminate the operators from the hypothesis; the introduction axioms introduce the operators into the consequence.

Elimination Axioms

And Elimination
Axiom 1. $p \wedge q \vdash p$
Axiom 2. $p \wedge q \vdash q$

Or Elimination
Axiom 4. $p \vee q, {\sim}q \vdash q$
Axiom 5. $p \vee q, {\sim}q \vdash p$

Not Elimination
Axiom 8. ${\sim}p \supset \rightarrow\!\leftarrow \vdash p$

Hook Elimination
Axiom 10. $p, p \supset q \vdash q$

Introduction Axioms

And Introduction
Axiom 3. $p, q \vdash p \wedge q$

Or Introduction
Axiom 6. $p \vdash p \vee q$
Axiom 7. $q \vdash p \vee q$

Not Introduction
Axiom 9. $p \supset \rightarrow\!\leftarrow \vdash {\sim}p$

Rules of Inference. We have three rules of inference (We use $\Rightarrow$ for if ... then ... in our metalanguage).

Conditional Proof

Rule 1. $\Gamma, p \vdash q \Rightarrow \Gamma \vdash p \supset q$

This rule can be thought of as an introduction rule, the counterpart to Axiom 10. Conditional proof says that if we know that a set of formulas Γ unioned with p implies q, then we know that Γ without p implies that if p is true then q is true.

Additional Assumptions

Rule 2. $\Gamma \vdash p \Rightarrow \Gamma,\Delta \vdash p$

This structural rule says that if a set of formulas Γ syntactically imply p, then Γ together with the formulas in Δ still syntactically imply p. Notice that this rule implies that if a set of formulas Γ syntactically imply p, then we can add from one to an infinite number of formulas and still syntactically imply p.

Consequences as Assumptions

Rule 3. $\Gamma \vdash p$ and $\Gamma, p \vdash q \Rightarrow \Gamma \vdash q$

This structural rule says that if a consequence is proven from a set of formulas, then that consequence may subsequently be used as an assumption to prove other formulas.

These last two rules of inference are intuitively assumed in the proofs beginning logic students construct.

EXAMPLE PROOF

Here is an example of how the proofs constructed in beginning and intermediate logic courses would be written in our truth-functional system.

Example Proof: The following is a proof of what is usually called hypothetical syllogism, $p \supset q, q \supset r \vdash p \supset r$.

BEGINNING LOGIC PROOF	
1. $p \supset q$	Premise
2. $q \supset r$	Premise
3. p	Assumed Premise
4. q	Modus Ponens 1, 3
5. r	Modus Ponens 2, 4
6. $p \supset r$	Conditional Proof 3-5

PROOF IN OUR SYSTEM	
1. $p \supset q, p \vdash q$	*Hook Elim.*
2. $p \supset q, p, q \supset r \vdash q$	*Adtl. Ass.* 1
3. $q, q \supset r \vdash r$	*Hook Elim.*
4. $p \supset q, p, q \supset r , q \vdash r$	*Adtl. Ass.* 3
5. $p \supset q, q \supset r , p \vdash r$	*Con. Ass.* 2, 4
6. $p \supset q, q \supset r \vdash p \supset r$	*Condtl. Pf.* 5

THEOREMS

As a result of our definition of syntactical implication, we can show by induction over implication (much like induction over the numbers or well-formed formulas) that our system is both sound and compact. To prove these two desired properties for our truth-functional system, we must show that 1) the axioms of our system have the desired property and 2) the rules of inference preserve the desired property. We can also show that our system is effectively enumerable and complete. Note that we are constructing metatheorems for soundness, compactness, effective enumerability, and completeness. That is, we are constructing proofs about our proof system outside of the truth-functional system we have created. Notice also that each of the axioms and rules of inference are required at least once in the following theorems.

SOUNDNESS

$\Gamma \vdash \sigma \Rightarrow \Gamma \models \sigma$

Soundness says that if a set of formulas, Γ, syntactically implies a formula σ, then Γ semantically implies σ.

1) We first prove that each of the axioms are sound.

Proof for Axiom 1. $p, q \vdash p \wedge q$

Assume that ν satisfies p, q. By Definition 3, this means $\nu(p)$=T and $\nu(q)$=T which by Definition 2 implies $\nu(p \wedge q)$=T, which by Definition 3 means ν satisfies $p \wedge q$. Therefore, by Definition 4, $p,q \models p \wedge q$.

Proof for Axiom 10. $p, p \supset q \vdash q$

Assume that ν satisfies $p, p \supset q$. By Definition 3, $\nu(p)$=T and $\nu(p \supset q)$=T. By Definition 2, $\nu(p)$=F or $\nu(q)$=T. Since $\nu(p)$=T, it must be the case that $\nu(q)$=T. By Definition 3, ν satisfies q. Thus, by Definition 4, $p, p \supset q \vdash q$.

These two proofs are typical of how the proofs of soundness for the other axioms proceed.

2) We now prove that the three rules of inference preserve soundness. For each rule of inference, we assume that the hypothesis is sound and show that the consequence is sound.

Proof for Rule 1. $\Gamma, p \vdash q \Rightarrow \Gamma \vdash p \supset q$

Assume $\Gamma, p \models q$. Using Definition 4, if ν satisfies Γ, p then ν satisfies q. Assume ν satisfies Γ. Show ν satisfies $p \supset q$. We use proof by cases. Case 1: $\nu(p)$=F. By Definition 2, $\nu(p \supset q)$=T and thus by Definition 3, ν satisfies $p \supset q$. Case 2: $\nu(p)$=T. Using Definition 3, ν satisfies p, so ν satisfies Γ, p. Together with our initial assumption, this implies ν satisfies q. By Definition 3, $\nu(q)$=T which implies by Definition 2 that $\nu(p \supset q)$=T and thus by Definition 3, ν satisfies $p \supset q$. Hence, by Definition 4, $\Gamma \models p \supset q$.

Proof for Rule 2. $\Gamma \vdash p \Rightarrow \Gamma, \Delta \vdash p$

Assume $\Gamma \models p$. By Definition 4, if ν satisfies Γ then ν satisfies p. Assume ν satisfies Γ, Δ. The fact that ν satisfies Γ, together with our initial assumption implies that ν satisfies p. Therefore, by Definition 4, $\Gamma, \Delta \models p$.

Proof for Rule 3. $\Gamma \vdash p$ and $\Gamma, p \vdash q \Rightarrow \Gamma \vdash q$

Assume $\Gamma \models p$ and $\Gamma, p \models q$. By Definition 4, 1) if ν satisfies Γ then ν satisfies p and 2) if ν satisfies Γ, p then ν satisfies q. Assume ν satisfies Γ. By 1), ν satisfies p, so ν satisfies Γ,p. Together with 2), this implies ν satisfies q. Thus, by Definition 4, $\Gamma \models q$.

COMPACTNESS

$\Gamma \vdash \sigma \Rightarrow \Gamma^{o} \vdash \sigma$, where Γ^{o} is a finite subset of Γ

Compactness says that if a set of formulas Γ implies a formula σ, then some finite subset of Γ implies σ. Notice that axiomatic systems of truth-functional logic which use modus ponens as the one and only rule of inference trivially satisfy compactness, since the hypothesis of modus ponens is finite. However, compactness is not automatic for our system, since the hypotheses of our rules of inference can be infinite sets of formulas.

1) Compactness is automatically proven for the axioms since the hypothesis of each axiom is finite.
2) We now prove that the three rules of inference preserve compactness. For each rule of inference, we assume that the hypothesis is compact and show that the consequence is compact.

Proof for Rule 1. $\Gamma, p \vdash q \Rightarrow \Gamma \vdash p \supset q$

Assume Γ,p $\vdash$q is compact. Then there exists a Γ^{o} which is a finite subset of Γ,p such that $\Gamma^{o} \vdash$q. Γ^{o}-{p},p $\vdash$q since if p is already in Γ^{o} then Γ^{o}-{p},p simply equals Γ^{o}, which implies q, and if Γ^{o} does not contain p then Γ^{o}-{p},p equals Γ^{o},p, which implies q by by *additional assumptions*. By *conditional proof*, Γ^{o}-{p} $\vdash p\supset q$, where Γ^{o}-{p} is a finite subset of Γ, as required.

Proof for Rule 2. $\Gamma \vdash p \Rightarrow \Gamma, \Delta \vdash p$

Assume $\Gamma \vdash$p is compact. Then there exists a Γ^{o} which is a finite subset of Γ such that $\Gamma^{o} \vdash$p. Note that Γ^{o} is a finite subset of Γ,Δ.

Proof for Rule 3. $\Gamma \vdash p$ *and* $\Gamma, p \vdash q \Rightarrow \Gamma \vdash q$

Assume $\Gamma \vdash$p and Γ, p $\vdash$q are compact. Then there exists a Γ^{o} which is a finite subset of Γ such that $\Gamma^{o} \vdash$p and a Γ^{*} which is a finite subset of Γ, p such that $\Gamma^{*} \vdash$q. By *additional assumptions*, Γ^{o},Γ^{*}-{p} $\vdash$p. We also know that Γ^{*}-{p},p $\vdash$q since if p is in Γ^{*}, then Γ^{*}-{p}, p simply equals Γ^{*}, which implies q, and if Γ^{*} does not contain p then Γ^{*}-{p}, p equals Γ^{*},p which implies q by *additional assumptions*. Using *additional assumptions* again, Γ^{o},Γ^{*}-{p},p $\vdash$q. Using *consequences as assumptions* , Γ^{o},Γ^{*}-{p} $\vdash$ q. Notice that Γ^{o},Γ^{*}-{p} is the union of finite subsets of Γ and is thus itself a finite subset of Γ.

EFFECTIVE ENUMERABILITY

We now show that our system is effectively enumerable. By *effectively enumerable*, we mean that there is some procedure which requires no ingenuity, has no randomness, and takes only a finite number of steps to list each of the consequences of our system. The idea behind effective enumerability is that there is a procedure to generate all of the consequences of our system.

Proof. In our system, each axiom has a countable number of instantiations. We know by compactness that we only need to consider a finite set of assumptions for our rules of inference. If a set of formulas does imply a consequence, then there will be a finite number of axioms and rules of inference which (by hook elimination or add introduction and then hook elimination) implies the consequence. That is, for each of the consequences of our system, we have a finite sequence of formulas which proves it. Since we can generate all possible sequences, we have a way of effectively ennumerating all of the consequences of our system.

COMPLETENESS

$\Gamma \vDash \sigma \;\Rightarrow\; \Gamma \vdash \sigma$

Completeness says that if Γ semantically implies σ, then Γ syntactically implies σ; that is, if σ is a valid consequence of Γ, then we can construct a proof using our axioms and rules of inference.

The proof is divided into 5 lemmas. 1) We first show that if any consistent set of formulas is satisfiable, then semantical implication implies syntactical implication. 2) We next show that we can find a maximal, consistent extension of a consistent set of formulas. 3) We then show that any maximal, consistent extension of a consistent set of formulas is what we call a truth-functional description of the world. 4) We next show that any truth-functional description of the world is satisfiable. 5) We finally show that any consistent set of formulas is satisfiable. Together, these steps imply that our system is complete.

USEFUL RESULTS

Before proving the 4 lemmas used to prove completeness, we include some results of our truth-functional system that will be useful in proving lemmas one and three. We define a set to be consistent if the set does not syntactically imply a contradiction. We also define a set to be maximal if the set contains every formula or its negation.

Result 0) All of the axioms can be generalized. That is, we can add a set of formulas to the assumptions of each axiom. For example, we can generalize *and elimination* : $\Gamma, \alpha \wedge \beta \vdash \alpha$.

Proof. We can add a set of formulas to the assumption of any axiom using *additional assumptions*.

Result 1) $\Gamma, \sim\alpha \vdash \rightarrow\leftarrow \;\Rightarrow\; \Gamma \vdash \alpha$ (Indirect Proof)

Proof. Assume $\Gamma, \sim\alpha \vdash \rightarrow\leftarrow$. By *conditional proof*, $\Gamma \vdash \sim\alpha \supset \rightarrow\leftarrow$. By generalized *hook elimination* $\Gamma, \sim\alpha \supset \rightarrow\leftarrow \;\vdash \alpha$. Using *consequences as assumptions*, $\Gamma \vdash \alpha$.

Result 2) $\Gamma \vdash \alpha$ and $\Gamma \vdash \sim\alpha \;\Rightarrow\; \Gamma \vdash \alpha \wedge \sim\alpha$

Proof. Assume $\Gamma \vdash \alpha$ and $\Gamma \vdash \sim\alpha$. Show $\Gamma \vdash \alpha\wedge\sim\alpha$. By generalized *and introduction*, $\Gamma,\alpha,\sim\alpha \vdash \alpha\wedge\sim\alpha$. Using *consequences as assumptions* twice, $\Gamma \vdash \alpha\wedge\sim\alpha$.

Result 3) $\alpha\in\Gamma \Rightarrow \Gamma\vdash\alpha$

Proof. Assume $\alpha\in\Gamma$. By generalized *and introduction*, $\Gamma, \alpha \vdash \alpha\wedge\alpha$. By generalized *and elimination*, $\Gamma,\alpha,\alpha\wedge\alpha \vdash \alpha$. Using *consequences as assumptions*, $\Gamma,\alpha \vdash \alpha$. Since $\alpha\in\Gamma$, $\Gamma,\alpha{=}\Gamma$. Hence, $\Gamma\vdash\alpha$.

Result 4) If Γ is maximal and consistent, then $\Gamma\vdash\alpha \Rightarrow \alpha\in\Gamma$

Proof. Assume $\Gamma\vdash\alpha$. Assume $\alpha\notin\Gamma$. By maximality, $\sim\alpha\in\Gamma$. By Result 2, $\Gamma\vdash \alpha\wedge\sim\alpha$, a contradiction of consistency. Therefore, $\alpha\in\Gamma$.

Result 5) $\Gamma,\sim\sim\alpha\vdash\alpha$ and $\Gamma,\alpha\vdash\sim\sim\alpha$

Proof. Using generalized *and introduction*, $\Gamma,\sim\sim\alpha,\sim\alpha \vdash \sim\sim\alpha\wedge\sim\alpha$. By generalized *conditional proof*, $\Gamma,\sim\sim\alpha \vdash \sim\alpha\supset\sim\sim\alpha\wedge\sim\alpha$. By generalized *not elimination*, $\Gamma,\sim\sim\alpha,\sim\alpha\supset\sim\sim\alpha\wedge\sim\alpha \vdash \alpha$. Hence, using *consequences as assumptions*, $\Gamma,\sim\sim\alpha\vdash\alpha$. The proof for $\Gamma,\alpha\vdash\sim\sim\alpha$ is similar and uses *not introduction.*

LEMMA 1

If all consistent sets of formulas are satisfiable, then for any Γ, $\Gamma\models\alpha \Rightarrow \Gamma\vdash\alpha$.

Proof. Assume all consistent sets of formulas are satisfiable and that $\Gamma|{=}\alpha$. Assume $\Gamma\nvdash\alpha$. Together with Result 1, this implies that $\Gamma,\sim\alpha \nvdash \rightarrowtail\!\!\leftarrow$. Hence $\Gamma,\sim\alpha$ is consistent and thus satisfiable. Hence $\Gamma\not\models\alpha$, which is a contradiction. Therefore $\Gamma\vdash\alpha$.

LEMMA 2

Given a consistent set of formulas Γ, we can find a maximal, consistent extension Γ^+ of Γ.

Proof: Let Γ be a consistent set of well-formed formulas. Let $\alpha_1 \ldots \alpha_n \ldots$ be an enumeration of all well-formed formulas. Construct Γ^+, an extension of Γ, as follows:

$$\Gamma^+_0 = \Gamma$$

$$\Gamma^+_{n+1} = \begin{cases} \Gamma^+_n, \alpha_{n+1} & \text{if } \Gamma^+_n, \alpha_{n+1} \text{ is consistent} \\ \Gamma^+_n, \sim\alpha_{n+1} & \text{otherwise} \end{cases}$$

$$\Gamma^+ = \bigcup \Gamma^+_i$$

Γ^+ is maximal; by construction, Γ^+ contains each formula or its negation. Γ^+ is consistent. If Γ^+ were not consistent, then by the compactness theorem, a finite subset of Γ^+, call it Γ^o, would necessarily imply a contradiction. This finite subset, which has a largest formula, α_i, implies the contradiction. Note that Γ^o is a subset of Γ^+_i for i. This implies some Γ^+_i implies a contradiction, which means Γ^+_i is not consistent. But by the way we have constructed Γ^+_i, it is consistent. Thus, Γ^+ must be consistent.

LEMMA 3

Any maximal, consistent extension of Γ is a truth-functional description of the world. That is, any maximal, consistent extension Γ^+ of Γ, satisfies the following conditions:

1) $(\sim\alpha)\in\Gamma^+$ iff $\alpha\notin\Gamma^+$
2) $(\alpha\wedge\beta)\in\Gamma^+$ iff $\alpha\in\Gamma^+$ and $\beta\in\Gamma^+$
3) $(\alpha\vee\beta)\in\Gamma^+$ iff $\alpha\in\Gamma^+$ or $\beta\in\Gamma^+$
4) $(\alpha\supset\beta)\in\Gamma^+$ iff if $\alpha\in\Gamma^+$ then $\beta\in\Gamma^+$

The proof proceeds by showing that Γ^+ satisfies each of the above conditions:

Proof of 1) $\sim\alpha\in\Gamma^+$ *iff* $\alpha\notin\Gamma^+$

$\Rightarrow$: Assume $\sim\alpha\in\Gamma^+$. Assume $\alpha\in\Gamma^+$. Using Result 3, $\Gamma^+\vdash\sim\alpha$ and $\Gamma^+\vdash\alpha$. Using Result 2, $\Gamma^+\vdash\alpha\wedge\sim\alpha$. But Γ^+ is consistent, so $\alpha\notin\Gamma^+$.

$\Leftarrow$: Assume $\alpha\notin\Gamma^+$. Show $\sim\alpha\in\Gamma^+$. By maximality, every formula or its negation is an element of Γ^+, so $\sim\alpha\in\Gamma^+$.

Proof of 2) $\alpha\wedge\beta\in\Gamma^+$ *iff* $\alpha\in\Gamma^+$ *and* $\beta\in\Gamma^+$

$\Rightarrow$: Assume $\alpha\wedge\beta\in\Gamma^+$. Assume $\alpha\notin\Gamma^+$ or $\beta\notin\Gamma^+$. By Result 4, $\Gamma^+\vdash\alpha\wedge\beta$. Case 1: $\alpha\notin\Gamma^+$. By maximality, $\sim\alpha\in\Gamma^+$, and thus using Result 3, $\Gamma^+\vdash\sim\alpha$. By *additional assumptions*, $\Gamma^+,\alpha\wedge\beta\vdash\sim\alpha$. Using generalized *and elimination*, $\Gamma^+,\alpha\wedge\beta\vdash\alpha$. Using *consequences as assumptions*, $\Gamma^+\vdash\alpha$. Using Result 2, $\Gamma^+\vdash\alpha\wedge\sim\alpha$, which contradicts the consistency of Γ^+. Case 2: $\beta\notin\Gamma^+$, is similar to case 1 and results in the same contradiction. Thus, $\alpha\in\Gamma^+$ and $\beta\in\Gamma^+$.

$\Leftarrow$: Assume $\alpha\in\Gamma^+$ and $\beta\in\Gamma^+$. Assume $\alpha\wedge\beta\notin\Gamma^+$. By maximality, $\sim(\alpha\wedge\beta)\in\Gamma^+$. Using Result 3, $\Gamma^+\vdash\alpha$ and $\Gamma^+\vdash\beta$. By the generalization of *and introduction*, $\Gamma^+,\alpha,\beta\vdash\alpha\wedge\beta$. Using *consequences as assumptions* twice, $\Gamma^+\vdash\alpha\wedge\beta$. Using Result 2, $\Gamma^+\vdash(\alpha\wedge\beta)\wedge\sim(\alpha\wedge\beta)$, a contradiction of consistency. Thus, $\alpha\wedge\beta\in\Gamma^+$.

Proof of 3) $\alpha\vee\beta\in\Gamma^+$ *iff* $\alpha\in\Gamma^+$ *or* $\beta\in\Gamma^+$

$\Rightarrow$: Assume $\alpha\vee\beta\in\Gamma^+$. Assume $\alpha\notin\Gamma^+$ and $\beta\notin\Gamma^+$. By Result 4, $\sim\alpha\in\Gamma^+$ and $\sim\beta\in\Gamma^+$. By Result 3, $\Gamma^+\vdash\alpha\vee\beta$, $\Gamma^+\vdash\sim\alpha$, and $\Gamma^+\vdash\sim\beta$. By generalized *or elimination*, $\Gamma^+,\sim\beta,\alpha\vee\beta,\sim\alpha\vdash\beta$. Using *consequences as assumptions* three times, $\Gamma^+\vdash\beta$. By Result 2, $\Gamma^+\vdash\beta\wedge\sim\beta$, a contradiction of consistency. Therefore, $\alpha\in\Gamma^+$ or $\beta\in\Gamma^+$.

$\Leftarrow$: Assume $\alpha\in\Gamma^+$ or $\beta\in\Gamma^+$. Show $\alpha\vee\beta\in\Gamma^+$. Case 1: $\alpha\in\Gamma^+$. Using Result 3, $\Gamma^+\vdash\alpha$. By generalized *or introduction*, $\Gamma^+,\alpha\vdash\alpha\vee\beta$. Using *consequences as assumptions*, $\Gamma^+\vdash\alpha\vee\beta$. Using Result 2, $\alpha\vee\beta\in\Gamma^+$. Case 2: $\beta\in\Gamma^+$, is similar and results in $\alpha\vee\beta\in\Gamma^+$. Therefore, $\alpha\vee\beta\in\Gamma^+$.

Proof of 4) $\alpha\supset\beta\in\Gamma^+$ *iff if* $\alpha\in\Gamma^+$ *then* $\beta\in\Gamma^+$

$\Rightarrow$: Assume $\alpha\supset\beta\in\Gamma^+$. Assume $\alpha\in\Gamma^+$. Show $\beta\in\Gamma^+$. By Result 3, $\Gamma^+\vdash\alpha\supset\beta$ and $\Gamma^+\vdash\alpha$. By generalized *hook elimination*, Γ^+, $\alpha\supset\beta,\alpha\vdash\beta$. Using *consequences as assumptions* twice, $\Gamma^+\vdash\beta$. Hence, using Result 3, $\beta\in\Gamma^+$. Therefore, if $\alpha\in\Gamma^+$ then $\beta\in\Gamma^+$.

$\Leftarrow$: Assume if $\alpha \in \Gamma^+$ then $\beta \in \Gamma^+$. This is the same as $\alpha \notin \Gamma^+$ or $\beta \in \Gamma^+$. Show $\alpha \supset \beta \in \Gamma^+$. Case 1: $\alpha \notin \Gamma^+$. By maximality, $\sim\alpha \in \Gamma^+$. By Result 3, $\Gamma^+ \vdash \sim\alpha$. By generalized *or introduction*, $\Gamma^+, \sim\alpha \vdash \sim\alpha \vee \beta$. Using *consequences as assumptions*, $\Gamma^+ \vdash \sim\alpha \vee \beta$. Using Result 5, $\Gamma^+, \alpha \vdash \sim\sim\alpha$. Using *additional assumptions*, $\Gamma^+, \sim\alpha \vee \beta, \alpha \vdash \sim\sim\alpha$. By generalized *or elimination*, $\Gamma^+, \sim\alpha \vee \beta, \sim\sim\alpha, \alpha \vdash \beta$. Using *consequences as assumptions*, $\Gamma^+, \sim\alpha \vee \beta, \alpha \vdash \beta$. By *conditional proof*, $\Gamma^+, \sim\alpha \vee \beta \vdash \alpha \supset \beta$. Using *consequences as assumptions*, $\Gamma^+ \vdash \alpha \supset \beta$. Case 2: $\beta \in \Gamma^+$, is similar and results in $\Gamma^+ \vdash \alpha \supset \beta$. Therefore, $\Gamma^+ \vdash \alpha \supset \beta$.

LEMMA 4

Any truth-functional description of the world is satisfiable. Hence, Γ^+, as defined above, is satisfiable.

Proof: We construct a truth assignment that satisfies Γ^+. We assign $v(\lambda)$=T for each atomic element $\lambda \in \Gamma^+$ and $v(\lambda)$=F for each atomic element $\lambda \notin \Gamma^+$. The proof proceeds by induction over the formulas.

1) We first prove that the atomic elements satisfy Γ^+.

Proof. Given an atomic element λ, $\lambda \in \Gamma^+$ iff $v(\lambda)$=T by assignment.

2) We now prove that formulas built up using the rules of syntax satisfy Γ^+.

Proof for $\sim\alpha \in \Gamma^+$ iff $v(\sim\alpha)=T$

Assume $\alpha \in \Gamma^+$ iff $v(\alpha)$=T and $\beta \in \Gamma^+$ iff $v(\beta)$=T. By assumption, $\alpha \notin \Gamma^+$ iff $v(\alpha)$=F. By Lemma 3, $\sim\alpha \in \Gamma^+$ iff $\alpha \notin \Gamma^+$. By Definition 2, $v(\sim\alpha)$=T iff $v(\alpha)$=F. Hence $\sim\alpha \in \Gamma^+$ iff $v(\sim\alpha)$=T.

Proof for $\alpha \wedge \beta \in \Gamma^+$ iff $v(\alpha \wedge \beta)=T$

Assume $\alpha \in \Gamma^+$ iff $v(\alpha)$=T and $\beta \in \Gamma^+$ iff $v(\beta)$=T. By Lemma 3, $\alpha \in \Gamma^+$ and $\beta \in \Gamma^+$. By Definition 2, $v(\alpha \wedge \beta)$=T iff $v(\alpha)$=T and $v(\beta)$=T. Hence $\alpha \wedge \beta \in \Gamma^+$ iff $v(\alpha \wedge \beta)$=T.

The proofs for $\alpha \vee \beta \in \Gamma^+$ iff $v(\alpha \vee \beta)$=T and $\alpha \supset \beta \in \Gamma^+$ iff $v(\alpha \supset \beta)$=T proceed similarly.

LEMMA 5

All consistent sets of formulas are satisfiable.

Proof. Assume Γ is a consistent set of formulas. By Lemma 2, there is a maximal, consistent extension Γ^+ of Γ. By Lemma 3, Γ^+ is a truth-functional description of the world. By Lemma 4, Γ^+ is satisfiable. Since Γ is a subset of Γ^+, Γ itself is satisfiable. Thus all consistent sets of formulas are satisfiable. By Lemma 1, completeness follows.

As a final comment, note that a system similar to our truth-functional system could be constructed to include quantificational logic. We would simply need to add introduction and elimination rules for the universal and existential quantifiers.

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