Rejecting the Question: A Wittgensteinian Response to the Hale and Wright vs. Field Debate

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ONE CURRENT CONTROVERSY in the philosophy of mathematics involves the ontological status of numbers. The question is, simply put: do numbers exist as actual entities? There are multiple camps in this war, but the primary action involves a dispute between "Fregean" Platonism[1] and a form of contingent nominalism. Bob Hale and Crispin Wright have led the Platonist attack, with Hartry Field as their primary opponent. Wright and Hale suggest that the language and effectiveness of mathematics prove the necessary existence of numerical entities. Field counters that no such thing is proved; rather, the existence of numbers is contingent, and therefore all number-referring theories are untrue. Their dispute represents a widespread schism in the philosophy of mathematics: on one side, a reluctance to admit that the amazing effectiveness of our mathematical system could be due to anything short of its correspondence with real entities; on the other side, a reluctance to embrace abstract objects as necessarily real.

What concerns me here are not the details of the Hale and Wright vs. Field debate, but the schism which that debate represents. I would suggest that a Wittgensteinian approach to the problem resolves

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[1]Fregean Platonism is a position which maintains the existence of some abstract objects (e.g. numbers) without the traditional Platonic hierarchy of being.

the schism without falling prey to the most glaring weaknesses of either camp. Ultimately, we will see that resolving the question necessitates rejecting the question altogether.

Platonism and the Nominalist Response

Let us begin with the most obvious reason for a belief in the actual existence of numbers. Simply put, mathematics is tremendously useful. Our complicated mathematical system has supported our economy, chemistry, physics, and technology with absolute consistency—a remarkable feat. It seems, *prima facie*, that the best explanation for such utility is the assumption that numerals are existing objects, and that mathematics' correspondence with those objects accounts for its usefulness. Platonism is the default position, so to speak—if math can get us to the moon, then we have good reasons for believing it has an ontological basis. (Only a philosopher would think to ask otherwise.) From an intuitive point of view, the burden of proof falls on the person who wants to say that mathematics has no such basis.

To this challenge, nominalists have embarked on projects[2] to account for the major results of pure mathematics and science without referring to abstract entities (i.e., numbers) at all. Field's *Science Without Numbers* represents an impressive project in that vein, but he has not been alone. Charles S. Chihara has proposed that a "constructibility theory" can account for the "foundational tasks" of set theory in mathematics, to show that mathematics does *not* depend on the literal reality of sets. Arend Heyting's intuitionist logic attempts to establish a different kind of mathematics which avoids all reference to numbers.

None of these projects provides a tempting alternative to the system we already use, but all of them demonstrate the human capacity to correctly calculate without making reference to abstract entities. Their respective levels of success aside, we can conclude, at the very least, that a system of science or arithmetic which does not depend on a correspondence with abstract objects is at least plausible. The nominalist response to the utility argument, then, is that it is incorrect to insist

[2]For a critical analysis of these projects, see Chihara's *Constructibility and Mathematical Existence*.

that mathematics must be based on a numerical ontology simply because it works. Non-referential mathematics also work (even if less effectively).

The debate is not yet resolved, however. The Platonists will quickly respond that even if these non-referential systems were just as useful as our standard number-based systems, they would still not *disprove* the existence of numbers. The fact that mathematics can be done without reference to numbers does not mean that they don't exist—merely that we do not have to refer to them.

To buffer their argument, Hale and Wright point out that there is additional independent evidence to support the ontological status of numbers. Fregean Platonism can be established, they claim, by an analysis of our mathematical language. The idea is basically that "objects are what singular terms, in their most basic use, are apt to stand for" (Wright 73). 'Number' is used as a singular term in true arithmetical statements, and therefore refers to objects. Or, to put it another way, the conceptually true statements we make about numbers imply that numbers actually exist.[3] We can say, for example: "The number of Fs is identical to the number of Gs if and only if there are exactly as many Fs as Gs;" and that "there are exactly as many Fs as Fs". These two logically true statements imply that "the number of Fs is identical to the number of Fs". On the Russellian assumption that a statement can only be true if all its terms denote, Hale and Wright infer "the number of Fs exists" (Tennant 311).

Briefly, "the notion of a singular term can be characterized syntactically and that in light of that characterization numerical expressions do indeed function as singular terms in arithmetical statements, whose truth demands, accordingly, the existence of numbers as objects" (Wright 78). This argument leaves the nominalist in the uncomfortable position of having to either deny that we use numerical statements as referents, or to deny that the numerical statements are true.

[3]The issue is more complex than I have stated it, but the complexities, unfortunately, do little to settle the argument between nominalism and Platonism. For the purposes of this paper, I have decided to deal with the syntactic argument in its simplest possible form. For the complex version, see Field, "The Conceptual Contingency of Abstract Objects" and Hale and Wright, "Nominalism and the Contingency of Abstract Objects."

It is still not settled, however. Field expresses overall discomfort with the idea that the implications of language can in any way prove things about the world. "The idea that the existence of numbers flows from the explanation of the concept of number is reminiscent of the ontological argument for the existence of God, according to which it follows from the very concept of God that God exists" (quoted in Wright 83). Field denies that the argument above proves anything about reality at all. We have to already assume that numbers exist in order to say "the number of Fs equals the number of Gs if and only if there are exactly as many Fs as Gs." So, in fact, what is being stated is that: "the number of Fs is identical to the number of Gs *if* (there are exactly as many Fs as Gs and *the number of Fs exists* and/or *the number of Gs exists*" (Field 287). The syntactical argument proves nothing one way or the other, and there remains no external evidence for the existence of numbers.

Nominalists will further point out that even if we are convinced by the utility argument or the syntactic argument that numbers do exist, we still have no way of knowing how or if we encounter them. By definition, we cannot perceive abstract objects; we cannot verify in any usual way what the number entities are or what corresponds to them.[4] We still have to ask, for example, "what is 2?" Appealing to abstract entities does not answer the question anyway. From these considerations, Field draws the conclusion that there is no way to assert either the necessary existence nor the necessary non-existence of numbers. Rather, their existence is *contingent*. They might have existed, or, without changing our language or mathematical systems, they might not have existed. Since there is no independent evidence for the existence of numbers, and we cannot perceive them anyway, we can safely conclude that numbers do not in fact exist.

Platonists will be quick to point out a serious weakness with the contingency theory: it does not, ultimately, explain anything. To say that the existence of abstract objects is contingent merely raises the

[4]Paul Benacceraf offers a particularly strong verifiability argument. One person claims that 3 belongs to 17 because the numbers progress as follows: {0},{0,{0}},{0,{0},{0}}}. The other claims that 3 doesn't belong to 17 because the numbers progress thus: {0},{{0}},{{{0}}}. The question is, how do they resolve their difference? To what criteria do they appeal to find the "right" answer?

question: "On what is the existence of numbers contingent?" Field does not answer that question; in fact, it is hard to imagine what kind of an answer could possibly be given (Hale and Wright 114–5). So the contingency argument does not harm the Platonic position. Also, utility of mathematics deserves some explanation, and the Fieldian approach rejects the Platonic explanation without offering another.

Thus the debate rages on. There are compelling arguments offered by both sides, and compelling criticisms of those arguments, but nothing from either camp can be considered absolutely conclusive. As the debate stands, the only reason for choosing one side or the other is a taste or distaste for the idea of abstract entities.

Rejecting the Question

If neither side can conclusively prove their point, we are forced to explore the possibility of seeking not just a new argument, but a different view of what mathematics is. We may need to stop looking at mathematics as purely a system of numerals which may or may not correspond to facts. We need to consider the possibility that mathematics is not something about which we can even ask the platonist/nominalist question.

Ludwig Wittgenstein offers a compelling possibility for such an alternative view. His mathematics is not a system of correspondence or non-correspondence, but is a component of our relationship with reality. At the most fundamental level, he believes that mathematics provides the structure through which we understand our world; a "framework for description" (*Remarks* 356). Without mathematics we would not be able to understand and question our world with the sophistication in which we do. "Mathematics—I want to say—teaches you, not just the answer to a question, but a whole language-game with questions and answers" (381). He refutes the assumption that useful calculations must be based on empirical facts by asking "why should it not rather determine what empirical facts are?" (383) In fact, even where it seems that a mathematical proposition refers to an external reality, it is only the "acceptance of a new measure of reality" (162). Mathematics is not an ontologically based method for solving problems we encounter in the world—through the framework of mathematics, we determine which problems need to be solved.

On this view, there is no need to suppose that numerals refer to actual number-objects in some alternative reality. "They 'treat' of nothing" (*Tractatus* 6.124). Though mathematics forms a framework for describing the empirical world, its propositions are not themselves empirical.[5] They are grammatical—they tell us in what manner it is appropriate to transform one sign or proposition into another. Mathematical statements are not descriptions of an abstract reality, but propositions "of *grammar* concerning the transformation of signs" (*Remarks* 169, my emphasis). Mathematics, for Wittgenstein, is primarily a technique, a process, a series of rules guiding an extremely useful "language game." Our world is significantly determined by the mathematics we have developed, and not the other way around. From Wittgenstein's point of view, then, the question as to whether or not numbers exist as abstract objects is a senseless one. Our system of mathematics is simply that: a system. It makes no more sense to question the ontological basis of a system of mathematics than to question the ontological basis of grammatical rules. There is no question of correspondence, there is simply how we do things.

Wittgenstein vs. Hale, Wright, and Field

Wittgenstein's approach to mathematics bears far more similarity to Field's nominalism than to any form of Platonism. However, unlike Field, Wittgenstein can reject the problematic aspects of Platonic theories without committing himself to a disbelief in number-referring theories. The following exploration of Wittgenstein's theory of mathematics will demonstrate how he manages to avoid the pitfalls of both camps.

By thinking of mathematics as a set of techniques rather than as a set of facts, Wittgenstein avoids the whole ontological problem. The

[5]No amount of empirical testing could prove the truthfulness of a mathematical proposition, for there would always remains the possibility that the next test will yield 5 apples from two sets of two apples. There is, surprisingly, no negative empirical test for mathematical propositions either. Wittgenstein noted, "If 2 apples and 2 apples add up to only 3 apples I don't say 'So after all 2+2 are not always 4;' but 'Somehow one must have gone'" (*Remarks* 97).

statement '2+2=4' does not, in his view, tell us anything about the number '2' nor the number '4'. The statement rather tells us the way to *use* the signs '+', '=', '2', and '4'. For if someone were to say that '2+2=5' we would not tell him that he was referring to the wrong entities; we simply say that he has added wrong. As Wittgenstein puts it, "the result is part of the calculation" (*Remarks* 79). That is, the result (4, in this case) is not something that is discovered through the addition of 2 and 2—it is a part of the grammatical sentence which begins "2+2=". Thus the statement shows us that the proper use of '+' between two '2's' entails a '4'. Likewise, the sign '=' tells us that we can substitute '2+2' for '4' and vice versa. But in neither case are we referring to entities or implying ontological relationships; we are simply following rules. There is nothing in this statement which demands an ontological explanation, so the debate about numerical ontology is simply superfluous.

None of this, of course, entails the necessary non-existence of numbers; Wittgenstein does not embrace the nominalist view. He asks, rather, whether abstract entities could provide a foundation for mathematics *at all*. Certainly, as discussed, there is a problem with trying to perceive objects which are by definition imperceptible. But suppose that through sophisticated abstract-object-detectors, or the birth of someone with particularly keen abstract-object-intuition, we were able to encounter abstract objects directly. Now suppose that we discovered that the object denoted by '12×12' is not, in fact, the number denoted by '144'. Wittgenstein remarks, "So everything worked out in this way is wrong!—But what does it matter? It does not matter at all!" (*Remarks* 90) That is, we would not discard our calculations and theories even if we were to discover that we were mistaken about the correspondence between our system of numbers and the ontological reality. We would still write '12×12=144', the way we always have. With that realization, it makes little sense to say that the truth of mathematics depends on its correspondence to real entities.

Additionally, even if we had access to numerical realities we would not necessarily know what to do with them. "What use is this knowledge [of the numbers] to me later on? I mean: how do I know what to do with this earlier knowledge when the step actually has to be taken?" (*Remarks* 36) My perception of the set of real numbers would not teach me how to do arithmetic or trigonometry or calculus; I still need to develop a set of techniques in order to be able to *apply* my perceptions. Even if there are abstract numerical objects, our perception of them is meaningless in the absence of rules, of conventions. If there is not agreement about "how we do things" with the numbers, then we cannot calculate. Period.

Clearly there are problems with the Fregean-Platonic account. But one may reasonably ask, on what grounds can something as inexorable as mathematics be seriously considered only rules and techniques? It is necessarily true that two and two are four and that three is the square root of nine—these statements are not up to scrutiny. Can we possibly, then, accept the nominalist conclusion that number-referring statements are not true, when we all speak and act and think as if they are? Platonism explains why we treat mathematics as inexorable; a viable alternative account of mathematics must do the same (and nominalism doesn't).

Here we must remember the fundamental focus of Wittgenstein's mathematical theory: we do not base our mathematical statements on the world, as the empiricists claim, but we base our world on our mathematical system. Once we have established our rules of transformation, the basics of our system, then within that system certain things are inexorable. "When we have determined one thing arbitrarily, something else is necessarily the case" (*Tractatus* 3.342). Having accepted the rule of addition, and having defined multiplication as multiple additions, then it must be the case that if 2+2+2=6 then 2×3=6. Our system demands that this be the case. Our conclusion is inexorable, not because it necessarily corresponds to actual entities, but because "this must come out if I proceed according to this rule" (*Remarks* 160). And we will all agree that in our system it is impossible for 2+2+2 not to yield the same result as 2×3. But this does not suggest that there is an object corresponding to '2×3'; it only proves that once we have accepted certain rules that certain conclusions *must* follow.

For calculation to work, then, it is vital that we all decide upon and follow the same rules. If there were disagreement about the results of '2+2' then "there isn't any calculating yet" (*Remarks* 356). In fact, if someone opted to follow different rules in calculating—insisting that 2+2=5, for example—then "we should declare him abnormal, and take no further account of his calculation" (79). But one may ask, why, if there is no further reality to appeal to, should we not allow others to calculate as they wish? Why shouldn't we honor the instincts of a person who "sees" math differently?

"The danger here," Wittgenstein answers, "is one of giving justification of our procedure where there is no such thing as a justification and we ought simply to [say]: *that's how we do it*" (*Remarks* 199). The deviant mathematician is just not following the rules that have been set up to govern calculation. And if he is not following the rules, then he is not calculating. Math has to be objective and inexorable, else *it would not be mathematics*. We do not need to appeal to an ontological realm here; it is clear that math is inexorable only because it would be useless to us if we did not act as if it were inexorable. We develop laws of mathematics, and "now it is *we* who are inexorable in applying those laws" (82). So the objectivity of mathematics is real; however, it is not due to its correspondence with abstract entities, but to *our* acceptance of rules.

Wittgenstein's rejection of Platonism seems thoroughly nominalist at this point, and may be read to imply a Fieldian rejection of the truth of mathematical statements along with an inability to account for the utility of mathematics. However, Wittgenstein's theory of mathematics as a system of grammatical rules must not be read as a total rejection of empiricism. On the contrary, by placing the inexorability of mathematics in the human realm, Wittgenstein provides a powerful account of the empirical basis of mathematics, and an explanation of its utility.

The rules and techniques which we have chosen to follow (and which we insist on following) are not chosen arbitrarily. "There correspond to our laws of logic very general facts of daily experience, [which] suggests the use of precisely these laws of inference," which we have chosen to use (*Remarks* 82). That is, we have specifically chosen to reason and to count and to calculate in a manner that produces useful results. Had our system of calculation failed to be useful we would have discarded it (see *Remarks* 373). Thus the utility of mathematics does not necessarily imply a numerical ontological foundation; if our numeral system frames our understanding of our world, the wonder would rather be if our system *lacked* utility. We have chosen this mathematics, this frame, precisely because it has proven useful. Wittgenstein explains:

> Counting is a technique that is employed daily in the most various operations of our lives. But is this counting only a *use* then; isn't

> there also some truth corresponding to this sequence? The *truth* is that counting has proved to pay. (Remarks 37–8)

There need not be anything corresponding to the rules of mathematics, besides the fact that those rules have proven extremely useful. All the more reason to follow those rules with exactness, then. Wittgenstein suggests that we could imagine alternative ways of counting or calculating if our world suggested the greater utility of another method (e.g., *Remarks* 91, 94). In that case, too, our mathematical system would have utility, because we would have decided on just that method of calculation which was useful. "It is the use outside of mathematics, and so the *meaning* of the signs, that makes the sign game into mathematics" (*Remarks* 257). Only in the context of useful applications can our rules for transforming signs achieve the status of a "mathematics"; again, it is mathematics not because of its correspondence but because of how it is used.

So Wittgenstein can answer the utility argument without sacrificing the objectivity of mathematics. The Hale-Wright argument from language attempts to force its opponents into choosing between the truth of mathematical statements and the language we use in making them. For Wittgenstein, as we have seen, there is no reason to deny that mathematical statements are true. We just need to be careful about what "true" means in the context of mathematical propositions. It can be true that 2+2 equals 4 in the sense that it could not equal 5; that's just how we do it. At the same time, it is not true that '2' and '4' refer to abstract entities. It may even be true, as Hale and Wright have argued, that arithmetical terms play the semantic role of singular terms which denote objects. But Wittgenstein would not be persuaded by such arguments. He would argue, to the contrary, that even if one could construct a sentence in which numerical terms function linguistically as object-referring terms, one has not proven that there is anything to which those terms refer (see *Investigations* 10). In fact, our attitude toward such terms indicates that we do not think of the referents as objects; if we did, we would be cautious about generalizing over them and would remain open to the possibility of empirical refutation. But we do not; we refuse the possibility of refutation, behaving quite unlike object-observers.

> Isn't it over-hasty to apply a proposition that one has tested on sticks and stones, to wavelengths of light? I mean, that

> $2 \times 5000 = 10000$. Does one actually count on it that what has proved true in so many cases must hold for these too? Or is it not rather that with the arithmetical assumption we have not committed ourselves at all? (*Remarks* 229)

We do not treat our mathematical statements as empirical statements about objects at all; thus we should not be misled by tricks of grammar into believing that they are. Wittgenstein denies that syntax can prove ontological facts, and thus has no reason to reject out-of-hand the statements and theories which seem to refer to number-objects. On the contrary, we need only guard against the mistaken philosophical interpretations of those theories which presuppose the existence of numbers. For example, Cantor's diagonal proof has been interpreted as showing that there are different cardinalities of infinity. On the assumption that numbers are ontologically real, this is a remarkable and puzzling result. From a Wittgensteinian view, however, this proof is no more philosophically interesting than the simplest results of arithmetic. If mathematics is a set of techniques, then all that Cantor has really shown is that "we have a method by which to upset any order" of real numbers (*Remarks* 133). He has not proven an empirical truth, but has demonstrated a new technique. Ultimately, the only "dangerous, deceptive thing about the idea: 'The real numbers cannot be arranged in a series' is that it makes the determination of a concept look like a fact of nature" (131).

That is, a mistaken interpretation of Cantor's proof makes what is in fact an interesting technique within the mathematical system look like a property of numbers themselves. It is precisely such mistaken interpretations which give rise to questions about ontology. As mathematicians, we need not be troubled by sentences like 'the real numbers cannot be arranged in a series' or 'all numbers have a successor'; these are statements which reflect our arithmetical system. We do not need to reject the truth of these statements in Field's way; the only trouble comes when philosophers determine these to be "true" sentences and look for the referents of 'number'. These statements do not refer because they are *not* statements of fact—they are *rules*. They are true without referring; they are true in the sense that they accurately reflect the rules or constraints on rules within our mathematical system. But it makes no sense to ask whether the rules themselves are true.

Conclusion

Wittgenstein approaches mathematics from a viewpoint not even considered by Hale, Wright, or Field. They have defined the ontological problem in terms of the contrast between a commitment to abstract objects and a rejection of the ultimate truth of number-based mathematical and scientific theories. Their debate forces one to choose, not so much between two possible explanations, but between two sets of undesirable philosophical baggage. Must we maintain the utility of mathematics only by supposing a foundation of objects we cannot directly encounter anyway? Must we avoid the problems of Platonism only by losing our faith in mathematics and science?

Wittgenstein offers an emphatic refusal, not of the possible answers, but of the question itself. It is ridiculous to pretend that mathematics is so useful because it is based on a realm with which we do not know how to communicate. It is equally ridiculous to pretend that mathematics is not true. The solution is to stop thinking of mathematics the way our language suggests, but to think of it instead as a grammar for interacting with the world. Wittgenstein's refusal of Platonism does not leave mathematics *sans* foundation, but founds it in the strongest possible realm. Mathematics is not about human behavior, or about forms of reality, but about the relationship between the two. It is in our interactions with reality, our actual behavior in the world, which (determine and) are determined by our system of mathematics. It is an influence on and a response to our needs in dealing with the world. It is useful to our relationship with reality because in so many ways it is our relationship to reality. In short, there is no reason we cannot have the best aspects of both philosophies.

Wittgenstein's response to the Hale and Wright vs. Field debate is to reject the parameters of the debate, and propose a new way to think about mathematical truth. This paper has not by any means resolved all the complexities of this issue, nor anticipated all possible objections to Wittgenstein's approach. Rather, I think it has shown that the current controversy cannot be resolved without a full-scale rejection of the controversy itself. Wittgenstein's alternative to the debate encloses the major attributes of our mathematical system, without being encumbered by the major difficulties of either side. This alone should make his philosophy more acceptable than either of the major positions presented here.

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