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Zeno's Paradox Revisited

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It seems obvious that objects can move. Most people would give you the Incredulous Stare if you claimed that they did not. But it is the sworn duty of all metaphysicians to doubt everything obvious. It is through this relentless process of doubt and questioning that we uncover hidden truths about reality that were not considered beforehand. Zeno of Elea doubted many seemingly apparent ideas about reality, including the very possibility of motion. While most contemporary metaphysicians and physicists generally agree on the existence of the possibility of motion, Zeno's insights continue to highlight fundamental conceptual gaps in our grasp of reality.

We will draw our attention to two of Zeno's paradoxes: the Achilles Paradox and the Dichotomy Paradox (Dowden 2024). Both rely on the assumption that space and time are infinitely divisible, meaning they are smooth and continuous. In the Achilles Paradox, a swift-footed Achilles races against a slow tortoise, who gets a head start. Though much faster, Achilles must first reach the tortoise's starting point at x_0. Achilles reaches x_0 after some time interval Δt_0. By the time he does, however, in that same time interval the tortoise has moved a certain distance $\Delta(x_1 - x_0)$ to reach a new point x_1. Achilles must now continue running to traverse $\Delta(x_1 - x_0)$,

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which takes him some time Δt_1. When he finally reaches x_1, the tortoise has now moved to x_2 during the time interval Δt_1. This process goes on *ad infinitum*. Zeno concludes that Achilles can never catch up to the tortoise.

In the Dichotomy Paradox, Achilles faces another challenge: to reach a finish line at a distance x away from him. But before reaching the finish line, he must traverse half the distance $x/2$. After he reaches $x/2$, he must travel half of his remaining distance towards the finish line at $3x/4$. Then, he must travel to half his remaining distance after $3x/4$, and so on *ad infinitum*. Consequently, Achilles never reaches the finish line. This paradox also works in reverse, where like before, he must make it to $x/2$. Before he can reach distance $x/2$, however, he must make it to the point $x/4$ and before he can make it to $x/4$, he must make it to half that distance at $x/8$ and so on *ad infinitum*. In this case, Achilles cannot make the first step!

Confronted with these evident contradictions, we are forced to reject one of the premises or assumptions of the argument. Zeno, following his mentor Parmenides, chose to reject the very possibility of motion. He argued that what we perceive as movement is an illusion, and the whole universe is a single, static, and motionless monad. This conclusion finds little support in contemporary thought, as changes in the state of physical entities relative to each other are so fundamental to the explanatory and predictive power in our experienced reality that it would be impractical and futile to deny it. However, the contradictions in these paradoxes remain and must be addressed.

The most widely accepted response to Zeno's paradoxes, known as the Standard Solution (Huggett 2018), hinges on mathematical properties taken to be inherent in space and time. We assumed for the paradoxes that space is a continuum, infinitely divisible. The continuum in mathematics is represented by the real numbers R. The properties of R are governed by the Axiom of Completeness. The axiom states: Every nonempty set in R that is bounded above has a least upper bound. One consequence of this axiom is the Density Theorem, which holds that for every $y \in R$ where $y > 0$, $\exists n \in N$ where $1/n < y$. Consequently, between any two real numbers with a finite difference, there exists an infinite number of real numbers between them (Abbott 2016, 19). For example, between 0 and 1, we find 0.1, and between 0 and 0.1, we find 0.01, and so on endlessly.

The Standard Solution relies on the property of density in R. In the case of Achilles, even though there are infinitely many points he must "pass through" between his starting point and the finish line (or the tortoise), his journey covers a finite distance. Due to the density of R, the infinite collection of distances can all be densely contained within the finite distance of the race, and the infinite collection of moments can all be densely contained within a finite time. Regardless of the infinite number of points he has to cross, Achilles *can* finish the race, because the distance traveled is ultimately finite, and by nature of *R* density can contain the whole infinitude of locations he has to pass through.

While compelling, the Standard Solution fails to fully resolve Zeno's paradox. The property of infinitely many points within a finite distance is already granted in the paradoxes' setup. Zeno likely did not have a full grasp of the mathematics of *R*, but his central concern—the paradoxes' troubling implications when applied to the physical world—remains unaddressed by the math. And while these contradictions might seem less glaring when applying the Axiom's continuum properties to the original paradoxes, they resurface in alternative formulations of the thought experiments (Huggett 2018).

Consider the Stacking Paradox: imagine a stack of an infinite number of blocks of alternating color, each one half as thick as the one below it. Despite the infinite number of blocks, the tower has a finite height, as the added thickness of all the blocks converges to a finite value. But what color would you expect to see at the top? Another intriguing configuration is the Lamp Paradox: a lamp is switched on, then off after half a minute, then back on after half that time, and so on, with infinite switches within a total of one minute. After the minute is up, the lamp must be either on or off, but which one would it be?

One particularly famous Zeno-Type paradox (and more specifically a Benardete Paradox) is the Grim Reaper Paradox, which has been used by numerous metaphysicians to support intervalism, the idea that the fundamental unit of time is intervals rather than points (Schmid 2023; Sieb 2019). Benardete paradoxes modify the setup to place an infinite set of items starting at a point and asymptotically approaching another point in the past. Each item satisfies some condition in case no prior item satisfies the same condition (Benardete). In this next example, a victim John is inconspicuously targeted by an infinite number of all-powerful grim reapers, who are all assigned to kill him. At 12:01, Grim Reaper 1 is assigned to kill John if no other Grim Reaper has killed him already. At 30 seconds prior to 12:01, Grim Reaper 2 is assigned to kill John if no other Grim Reaper has killed him already. With each increasing *n*, each Grim Reaper *n* is assigned to kill John half of the wait time of Grim Reaper *n* + 1 after noon if nobody else has killed John. This goes on *ad infinitum* until 12:00, where there is no assigned grim reaper, leading to the question: which reaper claims John's life? If we assume that there is a Grim Reaper *m* that does kill John, then we would know that there is some Grim Reaper *m* + 1 assigned to kill John half the time after 12:00 before Grim Reaper *m* was assigned to kill him. This means that

Grim Reaper $m + 1$ would have been the one to kill John, and not Grim Reaper m, giving us a contradiction.

The Standard Solution, while offering a partial resolution for Zeno's original paradoxes, fails to address the challenges posed by these reformulations. In the Stacking Paradox, a tower of blocks stands with each block half the thickness of the one below, resulting in a finite stack height but an infinite number of blocks. The Axiom of Completeness allows for this infinite number within a finite space. However, a contradiction is apparent: if there's an infinite number of blocks, there can be no "top" *final* block and thus no color on top, yet one would be able to clearly see the top of the finite tower. The Lamp Paradox presents a similar conundrum. After a minute of infinite switching, the lamp must be either on or off. However, the Axiom of Completeness suggests there can be no final switch, as the process of switching continues infinitely: another contradiction. Finally, the Grim Reaper Paradox reaches issues akin to the others. The Axiom dictates that no Grim Reaper can definitively kill John since infinity requires that there never be a Grim Reaper with the *earliest* time requirement. But, there is no possibility of John surviving, as each reaper is tasked to kill him if everyone with a prior assignment fails, and yet there is nothing to stop each prior grim reaper from killing John before the next one.

While these paradoxes showcase the inadequacy of the Standard Solution, a physicist would not take any of these thought experiments seriously. Each of these thought experiments contains premises in their setup that a physicist would quickly reject on the basis that they blatantly violate the accepted laws of physics, rendering the whole thought experiment pointless. The Lamp Paradox requires flipping the switch faster than the speed of light, the Stacking Paradox demands that blocks be eventually thinner than a single atom, and unfortunately for the Grim Reaper Paradox, there is not any empirical evidence of the existence of Grim Reapers. If a formulation of a Zeno-type paradox is to pose significant issues to our current model of reality, it needs to conform to all aspects of this model, which requires that this thought experiment obey all of the currently accepted laws of physics.

Here I present a new Benardete Paradox thought experiment where all factors conform with the current accepted model of the laws of physics, specifically adhering to General Relativity, Quantum Mechanics, and the Standard Model of Particle Physics. This thought experiment is therefore a *possible* (though unlikely) occurrence that satisfies Zeno's assumptions of his original thought experiments. Even with such a formulation, we still arrive at a contradiction.

Let there exist some "wall" or plane of photons extending infinitely along the *x* and *y* spatial dimensions. This wall is moving at the speed of light (*c*) in the *z* direction approaching an empty parallel plane. At some time *t*, the wall of photons will have just passed through this empty plane. Note that *t* is *not* when the photon wall is *at* the plane, but rather instantly beyond it. One centimeter away in the *z* direction from the empty plane is an electron neutrino that we will denote ν_1. Half a centimeter in the *z* direction away from the first neutrino (and towards the empty plane) is a second neutrino that we will denote ν_2. At decreasing *z* distance intervals, we have neutrinos denoted ν_3, ν_4, ν_5, ... *ad infinitum*. Each neutrino ν_n is half the *z* distance away from the empty plane as neutrino ν_{n-1}. These neutrinos of increasing number notation asymptotically approach the empty plane, but no single neutrino will be on or past the plane. This is allowed through the accepted notion in both Quantum Mechanics and General Relativity that spacetime is a continuum and therefore adheres to the density consequences of the Axiom of Completeness. Since space is infinitely divisible, and neutrinos are defined as point masses, a neutrino can be located at any distance $z \in \mathrm{R}$ from the empty plane, permitting this asymptotic setup.

To avoid concerns arising from the Uncertainty Principle, we assert that these neutrinos will be arranged in their asymptotic positions at time *t*, at the time of interaction with the photon plane. At no other time are the locations of the neutrinos relevant. To ensure a cohesive interaction, we further stipulate that these neutrinos are entangled, meaning their states are interconnected. This entanglement dictates that when one neutrino interacts with the photon wall, the locations of all other neutrinos become instantaneously known at *t*. We will not inquire about the neutrinos' momenta at any given time within this experiment and only require the information of their locations at *t*. This allows us to distinguish and denote individual neutrinos within the arrangement.

To prevent gravitational interference, we assume the neutrinos are dispersed along the *x* and *y* directions ensuring they do not concentrate enough to form a black hole. The specific *x* and *y* coordinates of each individual neutrino are irrelevant as long as this dispersion condition is met. See Figure 1 for a visual representation of this setup.

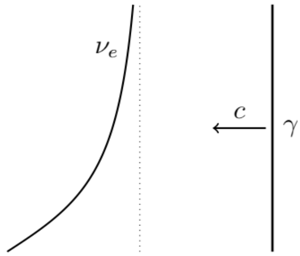

Figure 1: The photon plane is represented by the line γ moving at speed *c*. The dotted line represents the empty plane. Along the curve ν_e, are the locations of the neutrinos asymptotically approaching the empty plane. Their positions are where they will be at time *t* when γ instantly crosses the empty plane. Up and down is the *x* dimension and left and right is the *z* dimension.

As the photon wall passes through the empty plane at time *t*, with which neutrino will the photon plane interact? Or to express it in simpler terms, with which neutrino will the photon wall interact "first"? Since space and time are assumed to be continuous, the photon plane will not interact with multiple neutrinos at once, so each neutrino has a distinct *z* distance value. Proof by contradiction demonstrates that attempting to give an answer to the question is problematic:

1. Assume the setup of the thought experiment as described above.

2. Let the neutrino denoted ν_n be the one with which the photon plane interacts at *t*.

3. By 1: The neutrinos are arranged such that there are an infinite number of neutrinos getting progressively closer to the empty plane with increasing numerical denotation.

4. By 3 and 2: There must exist some ν_{n+1} such that it is closer to the empty plane than ν_n.

5. By 4 and 1: The photon plane would have interacted with ν_{n+1} before interacting with ν_n.

6. By 5: The neutrino ν_n is not the photon with which the photon plane interacts at *t*, contradicting 2. ⊥

This setup leads us to a contradiction, rendering this situation described by the paradox to be logically prohibited and, therefore, impossible. However, significantly, the entirety of the setup conforms to the current model of all accepted laws of physics and the Axiom of Completeness (preventing the use of the Standard Solution as a resolution). To resolve this paradox and avoid contradiction, we are required by the logical rules of our language to deny one of the premises or underlying assumptions. All of the premises and assumptions are part of the contemporary model of physics assumed in the setup. This model, as mentioned earlier, conforms with all accepted laws of physics according to General Relativity, Quantum Mechanics, and the Standard Model of particle physics. But we are logically required to reject some condition required by these theories. We are left to make a difficult choice, and we must be careful in our decision.

It would be counterproductive to reject notions of any of these theories that have significant direct empirical evidence to back them up. When considering all cases within epistemological possibility, it is possible that some factor of a theory with major empirical evidence in its support could be incorrect and that the supporting empirical phenomena are misrepresented. Considering that both Quantum Mechanics and General Relativity are the two most successful physical theories in terms of physical evidence in their support, however, this is significantly unlikely and would require copious rewriting of physics from scratch. The Standard Model of Particle Physics is also a highly successful theory in terms of empirical backing. While reforming these physical theories in some way is required, the most pragmatic option would be to first consider the notions of the theories that have no direct empirical support.

A promising avenue for resolving the paradox is questioning the application of the Axiom of Completeness to spacetime. Note that this is not rejecting the Axiom itself, as it is a very useful axiom in mathematics that has brought about significant success in various fields. However, valid and powerful mathematical principles have been applied to the physical world incorrectly before. Therefore, I propose to reject the notion that the Axiom of Completeness should be applied to spacetime. This rejection can be realized through quantizing space or by quantizing time. Both General Relativity and Quantum Mechanics assume both space and time to be a continuum. Thus, these theories must be modified to operate with such quantization. Quantum Mechanics defines space and time to be two distinct entities, making the theory ultimately agnostic as to which one should be quantized. However, General Relativity defines space and time to be both part of the same structure: spacetime. The most pragmatic option to work into General Relativity would be quantizing both space and time. This approach holds significant potential as it avoids abandoning the well-established theories of General Relativity and Quantum Mechanics, and rather simply demands a more fundamental theory that approximates to these two.

Far from being a new idea, quantizing spacetime stretches back to the 1930s, championed by physicists like Matvei Bronstein (140). He argued that consistently reconciling Quantum Mechanics and General Relativity necessitated rejecting the continuous nature of spacetime for a granular, quantized one. This concept is the central assumption of Loop Quantum Gravity, a vibrant research area spearheaded and under active research by physicists like Carlo Rovelli (225).

This quantization follows the heart of the metaphysical concept of indivisible units first proposed by Democritus. Imagine dividing space and time into ever-smaller segments until reaching a fundamental unit, an "atom of space" as Carlo Rovelli calls it (170). This indivisible spatial unit would define the minimum length and volume possible, similar to how atoms define the limits of matter's divisibility. When a particle is in a specific location, instead of being a point in three-dimensional space R^3, it occupies an entire spatial unit. Likewise, time would be broken down into discrete quanta, replacing the continuous flow with distinct "ticks" or intervals of duration. These intervals have distinct lengths in duration. Loop Quantum Gravity (LQG) posits that these fundamental units are the Plank Length, Plank Volume, and Plank Time, incredibly tiny scales representing the limits of our measuring capabilities.

In this quantized picture, motion becomes discrete as well. Picture a particle occupying a single "atom of space" at one time interval. In the next interval, it can only hop to an adjacent space unit, never skipping over units. Particles making such jumps at every interval would be moving at the speed of light. Conversely, taking multiple intervals to reach an adjacent unit represents slower-than-light motion (see Figure 2). While this is a simplification of how Loop Quantum Gravity describes motion and time, the basic principles are the same.

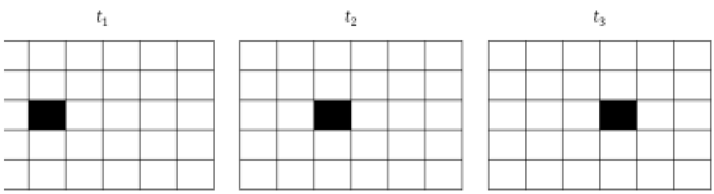

Figure 2: Particle moving at the speed of light. The particle is represented by the black shade filling in one unit of space. As each time interval passes, it moves over to an adjacent space. Note that quantized space is not necessarily cubic as shown here.

If we apply this quantization of spacetime to Zeno's original paradoxes, we find resolutions. When Achilles rushes to the finish line, he reaches a point where only one spatial unit separates him from victory. Unlike a continuum, he does not need to traverse half that unit. Instead, at the next time interval dictated by his speed, he'll simply occupy that final space quantum. In the following interval, he'll cross the line. Similarly, at the start of the race, he wouldn't need to move halfway into the initial unit but simply hop to the adjacent one at the appropriate time interval.

Let us extend this method to the race with the tortoise. As Achilles approaches, he again reaches a point where only one "atom of space" separates them. Due to his speed, he'll hop to the next unit at an earlier time interval than the tortoise, allowing him to finally overtake. The time intervals between jumps are determined by each participant's speed, ensuring the faster party covers more ground within the discrete framework. The other described Zeno-type paradoxes can be resolved in similar manners.

Crucially, the photon wall paradox finds a resolution within this framework. As the neutrinos approach the empty plane, their arrangement becomes constrained by the discrete nature of space. Eventually, some neutrino ν_n will occupy a spatial unit situated at a z distance of one unit away from the empty plane. Accordingly, ν_{n+1}, being next in the sequence, must then reside in the spatial unit directly adjacent to the empty plane. But where will ν_{n+2} be located? This arrangement eliminates the possibility of ν_{n+2} positioning itself halfway between the empty plane and ν_{n+1} along the z direction. The discrete nature of space forbids such intermediate locations. Consequently, all neutrinos subsequent to ν_n —collectively denoted as $\nu_{>n}$ —are forced to occupy spatial units directly adjacent to the empty plane. When the photon wall passes the empty plane at t, it interacts with all of these adjacent neutrinos simultaneously (see Figure 3). This coherent interaction between the photon wall and the neutrinos circumvents the paradox, as there's no longer a need to pinpoint a specific neutrino as the first to interact.

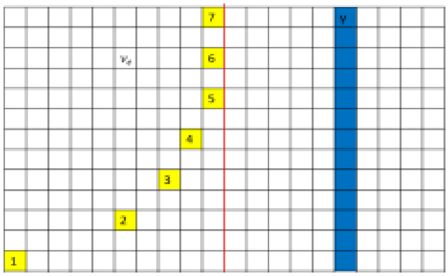

Figure 3: Resolution to the photon wall paradox using the quantization of space. The numbered spaces are the numbered neutrinos. The photon wall will interact with ν_5, ν_6, ν_7, . . . simultaneously.

While alternative logically equivalent solutions to the photon wall paradox might exist, the simplicity and naturalness of quantized spacetime make it a particularly attractive approach. Unlike some solutions that might introduce inconsistencies with established observations, quantizing spacetime aligns well with existing empirical data. Moreover, it finds strong support in rigorous physical theories like Loop Quantum Gravity. Notably, these theories are the only current physical models that offer non-contradictory resolutions to the paradox, making them the most promising avenues for navigating this thought experiment.

The implications of quantized spacetime extend far beyond resolving this specific paradox. It serves as a bridge between Quantum Mechanics and General Relativity, two pillars of modern physics that currently struggle to be reconciled. Exploring this intriguing framework might not only unlock solutions to other long-standing problems, but may also lead to entirely new predictions and a deeper understanding of the universe's fundamental structure.

By applying Zeno's approach to the physical world, paradoxes like the photon wall example reveal limitations in our current understanding of reality. This paradox exposes a flaw in the framework where accepted theories like General Relativity and Quantum Mechanics struggle to provide consistent explanations, even when all their rules are followed. Like all paradoxes, this challenge forces us to modify our model and explore alternative paradigms, or otherwise remain in ignorance.

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