Aristotle's Quantificational Logic

RYAN CHRISTENSEN

ECTHESIS (ἔκθεσις) IS A PROBLEMATIC ELEMENT of Aristotle's system of logic, as it is rarely used and never defined in his *Prior Analytics*. There has been some debate concerning its meaning, and there are currently three different interpretations. I shall argue in this essay that none of these interpretations is logically or textually adequate to explain the role of ecthesis in Aristotle's syllogistic. I shall further argue that his use of ecthesis indicates that Aristotle had a wide but undeveloped knowledge of quantificational logic.

The word ἔκθεσις means "setting forth" and is translated by Jenkinson as "exposition" and by Smith as "setting-out."[1] Aristotle uses the word in various ways. For example, in the *Poetics* he uses it to refer to Odysseus' "putting out" on the shores of Ithaca (1460a36; see Liddell, Scott s.v. ἔκθεσις). Within the context of Aristotle's logic, Patzig identifies three separate meanings of the term. The first meaning is used within the context of syllogistic proof and is the primary sense of interest in this essay. The second has to do with translating an argument in ordinary language into symbols. The third meaning is the opposite of the second and means illustrating a syllogistic mood by replacing its symbols by terms in ordinary language (Patzig 158).

---

[1]All my quotes from Aristotle are from Smith's translation, and all references to chapters are to those in *Prior Analytics* A. I will, however, use the word "ecthesis" in place of "setting-out" or any other word used to translate ἔκθεσις.

Ryan Christensen is a junior at Brigham Young University majoring in philosophy. This essay was awarded first place in the 2001 David H. Yarn Philosophical Essay Competition.

Despite these various senses, Aristotle uses the terms ἔκθεσις and ἔκθεσισθαι only once each in his account of syllogistic logic, at 29b14 and at 28a23, respectively (Smith, "Ecthesis" 113; Łukasiewicz 59). In his interpretation, Łukasiewicz counts three passages in which Aristotle gives an account of proof by ecthesis (59); Patzig identifies more, counting five different syllogistic moods that are proved by ecthesis (156–57). In his summary in chapter 7, Aristotle does not mention ecthesis. This leads Łukasiewicz to conclude that proof by ecthesis has "no importance for Aristotle's syllogistic as a system" (67), and leads Patzig to claim that Aristotle "accorded only a restricted value to ecthesis as a method of proof" (156). Smith's position might be interpreted as even more radical, for he entirely rejects the possibility of ecthetic proofs ("Ecthesis" 113).[2]

These two factors, ambiguity and scarcity of evidence, have contributed to the lack of understanding of ecthesis. There have been three major theories presented to explain ecthesis, but all of them fail to account for the evidence. I will briefly explain these theories and show why they are not satisfactory explanations. Finally, I will present my own interpretation and show why it fits the data better.

Łukasiewicz's Interpretation

Łukasiewicz's interpretation[3] of ecthetic proof is understood best in explanation of the passage in 30a4–14,[4] in which Aristotle proves the moods *Baroco* and *Bocardo*. Aristotle says:

[2]For Smith, all proofs require ecthesis, so an ecthetic proof is a matter of degree, not an issue for distinction.

[3]Łukasiewicz's interpretation is not original to Łukasiewicz, nor is he the only proponent of it. Anciently both Alexander and Galen held essentially the same interpretation, though Alexander saw it as referring only to some passages (Smith, "Ecthesis" 118). More recently, Patzig and Smith (in his more recent works) favor interpretations like Łukasiewicz's.

[4]Łukasiewicz doesn't actually defend his position by use of this passage, however. As Patzig says, "Unfortunately, [this] passage [is] not analyzed. In fact these proofs support Łukasiewicz's proposed exegesis far better than the passages he refers to" (161).

But in the middle figure, when the universal is affirmative and the particular is privative, and again in the third figure, when the universal is positive and the particular privative, the demonstration is not possible [through conversion]. Instead, it is necessary for us to set out [by ecthesis] that part to which each term does not belong and produce the deduction about this. (30a6–10)

Aristotle here rejects his usual means of proof, conversion and impossibility,[5] and says that ecthesis must be used to prove these moods. He does not, however, give any interpretation of how this proof is to be done.

An argument of the mood *Baroco* looks like this:

If A belongs to all B and A does not belong to some C, then B does not belong to some C.[6]

Following convention, with *a* representing "belongs to all," *i* representing "belongs to some," *e* representing "belongs to no," and *o* representing "does not belong to some," the above argument may be symbolized as follows:

A*a*B, A*o*C ⊢ B*o*C.

An alternate symbolization, making use of the symbols of modern quantificational logic, is as follows:

[5]Conversion is Aristotle's usual method of proof, by which he uses the first-figure syllogisms as paradigm cases. "Impossibility" is the word that Smith uses to translate *reductio ad absurdum*.

[6]I am here using the formulation developed by Łukasiewicz following Aristotle's language. For Aristotle, the predicate precedes the subject, much as in most systems of quantificational logic. This distinction is not essential for my argument. In traditional syntax, a syllogism of the mood *Baroco* looks like this:

All B are A

Some C are not A.

Therefore, some C are not B.

$$\forall x(Bx \rightarrow Ax)$$
$$\exists x(Cx \& {\sim} Ax)$$
$$\therefore \exists x(Cx \& {\sim} Bx).$$

Łukasiewicz's interpretation of ecthetic proofs depends on the following two inference rules:

> **L1.** If A belongs to some B, then there is a term C such that A and B belong to all C.
> **L2.** If A does not belong to some B, then there is a term C such that A belongs to no C and B belongs to all C. (Patzig 161)

Patzig later makes it clear that these are in fact equivalence rules, that the antecedent and consequent are interchangeable (161–62). Smith, following Łukasiewicz's own formulation of these laws into four separate statements (Łukasiewicz 61–62), symbolizes these laws as:

> **L1.** $AiB \vdash AaC, BaC$ (where C does not occur previously)
> **L2.** $AoB \vdash AeC, BaC$ (where C does not occur previously)
> **L3.** $AaC, BaC \vdash AiB$
> **L4.** $AeC, BaC \vdash AoB.$[7]

Patzig likewise gives symbolizations using quantificational logic (161).

In addition to these laws, an ecthetic proof for *Baroco* also requires the following conversion rule:

> **C.** $BeA \vdash AeB,$

as well as *Celarent*, a first-figure syllogism:

> *Celarent.* $AeB, BaC \vdash AeC.$

Using all these laws, an "ecthetic" proof for *Baroco* would look like this:

---

[7]These symbolizations are taken from Smith, "Introduction" *xxiv*, except that in them I have used "C" (where he uses "S") to be consistent with the nomenclature of Patzig's formulations.

1. A*a*B   (premise)
2. A*o*C   (premise)
3. A*e*N   2, **L2**
4. C*a*N   2, **L2**
5. N*e*A   3, **C**
6. N*e*B   1, 5, *Celarent*
7. B*e*N   6, **C**
8. B*o*C   4, 7, **L4** (conclusion).

Łukasiewicz's formal laws for explaining ecthetic proof work very well for explaining the proofs for *Baroco* and *Bocardo*.[8] They have difficulty, however, explaining the proof for *Darapti*, "the most important passage" in understanding ecthetic proof (Patzig 159). Aristotle's proof of *Darapti* says: "It is...possible to carry out the demonstration through [ecthesis]. For if both terms belong to every S, then if one of the S's is chosen (for instance N), then both P and R will belong to this; consequently P will belong to some R" (28a22–25). Following the letters used in the passage, *Darapti* may be symbolized as:

P*a*S, R*a*S ⊢ P*i*R.

In this case, Łukasiewicz's laws hold true only trivially, as **L3** is just *Darapti*. Therefore, a proof would go like this:

1. P*a*S   (premise)
2. R*a*S   (premise)
3. P*i*R   **L3** (conclusion).

But this is begging the question; the mood is proved by assuming it as an axiom. As Smith says,

> [**L3**] and [**L4**] seem...to be identical to *Darapti* and *Felapton*. Since Aristotle regards these as...in need of proof, then these rules

---

[8]For a proof of *Bocardo*, see Patzig 164–65, or Smith, "Introduction" *xxiv*. Neither of them gives a proof for *Baroco*, perhaps because it is slightly longer than the proof for *Bocardo*.

themselves would appear to be in need of justification [which Aristotle does not give]. He seems therefore open to a charge of circularity. ("Introduction" *xxiv–xxv*)

Smith also says that Alexander objected to this type of interpretation on grounds of circularity, "although of course not as an objection to Łukasiewicz" ("Ecthesis" 118).

This interpretation also fails to account for the text. Aristotle uses an ecthetic term N to prove the mood, something Łukasiewicz's theory fails to account for. As Smith says, "In this case Aristotle's introduction of a fourth term, N, in reducing *Darapti* is utterly inexplicable" ("Ecthesis" 118). Łukasiewicz sees no problem in this, saying, "It is of no consequence, of course, to denote this term by N rather than by C" (64). In his formulations of rules **L1–L4**, however, it is not N but S that he replaces by C. That is, Łukasiewicz can only account for three variables in this proof, while Aristotle clearly believes he needs four.

In sum, Łukasiewicz account of ecthetic proof explains well the passages on *Baroco* and *Bocardo* given in chapter 8, but it fails to explain other passages, such as the proof for *Darapti* in chapter 6.

Smiley's Interpretation

Łukasiewicz's failure can be seen as arising from his denial that Aristotle allows for singular terms (Łukasiewicz 60). Alexander, however, saw that there must be singular terms for logical as well as grammatical reasons (Smith, "Ecthesis" 119). Following this, an alternate interpretation has been given:[9]

> Ekthesis is similar to the use of free variables in modern systems of natural deduction. Having assumed that some a's are b, we are allowed to select an arbitrary particular instance of a, which is b.

[9]Smith credits this interpretation to Smiley, though Smiley does not work out a complete theory of ecthetic proofs. Smith follows Clark's version, which he believes is merely the extension of Smiley's theory of the syllogism. Lear also holds this view, and anciently Alexander explained some of the passages this way (Smith, "Ecthesis" 117–21).

This corresponds to existential instantiation in natural deduction. (Lear 4)

If we symbolize "B belongs to all A" as "$\forall x(Ax{\rightarrow}Bx)$" and "B does not belong to some A" as "$\exists x(Ax\&{\sim}Bx)$," we can easily prove *Bocardo*:

1. $\forall x(Bx{\rightarrow}Ax)$	(premise)
2. $\exists x(Cx\&{\sim}Ax)$	(premise)
3. $Cn\&{\sim}An$	2, existential instantiation
4. $Bn{\rightarrow}An$	1, universal instantiation
5. ${\sim}Bn$	3, 4, modus tollens
6. $Cn\&{\sim}Bn$	3, 5, conjunction
7. $\exists x(Cx\&{\sim}Bx)$	6, existential generalization (conclusion).

That is, Smiley's interpretation also can explain this proof, provided that we allow for other quantificational laws as well. This is a large proviso, and it is not accounted for in Smiley's interpretation.

This explanation, however, does not by itself adequately describe the ecthetic proof for *Darapti*, which in quantificational symbols is

$$\forall x(Sx{\rightarrow}Px)$$
$$\forall x(Sx{\rightarrow}Rx)$$
$$\therefore \exists x(Rx\&Px).$$

As it stands, this argument is invalid. For Aristotle, to say "B belongs to all A" is to imply the existence of A. That is, the statement "All A are B" implies "Some A are B." In order to solve a syllogism like *Darapti*, Smiley allows the inference

S1. $\forall x(Ax{\rightarrow}Bx)\vdash\exists x(Ax\&Bx)$,

and cites Church as his authority (Smiley 62, 68).

Church's metatheorem *331 states

$$\forall x\phi\vdash\exists x\phi,$$

which is a valid inference. It is not, however, equivalent to S1, and, of course, Church never claims that it is (Church 187). Not even Church

can infer "Some A are B" from "All A are B" without an additional premise that some A's exist.[10]

Taking, then, $\exists xSx$ as a suppressed premise, we can prove *Darapti* as follows:

1. $\forall x(Sx\rightarrow Px)$	(premise)
2. $\forall x(Sx\rightarrow Rx)$	(premise)
3. $\exists xSx$	(premise)
4. Sn	3, existential instantiation
5. $Sn\rightarrow Pn$	1, universal instantiation
6. $Sn\rightarrow Rn$	2, universal instantiation
7. Pn	4, 5, modus ponens
8. Rn	4, 6, modus ponens
9. $Rn\&Pn$	7, 8, conjunction
10. $\exists x(Rx\&Px)$	9, existential generalization (conclusion).

This proof, like that for *Bocardo* above, requires both existential instantiation and generalization, as well as universal instantiation.

Smiley's approach is much more plausible than Łukasiewicz's. However, it fails in a different way. Łukasiewicz's failure may be interpreted as stemming from his insistence that ecthesis (like all syllogistic statements) must be universal, while Smiley's error is his reliance on singular instances as being sufficient to explain ecthesis.

Smith's Interpretation

Because of these failures, Smith denies the existence of ecthetic proofs. Because the term ἔκθεσις is used in Greek mathematical proofs to refer to the letters used to label the various geometrical elements involved in the proof, Smith sees the term as having the same meaning in Aristotle's logic (Smith, "Ecthesis" 123–24). He says,

I argue for a rather different interpretation of ecthesis, not as a process appealed to (redundantly) in several isolated cases, but

---

[10]This is not, of course, unique to Church's system; nearly every textbook on quantificational logic will have a similar theorem.

rather as something present (at the metalogical level) in every one of Aristotle's deductions.... What Aristotle means by the term is...the "setting out" of the hypothesis of a theorem using letters. ("Ecthesis" 113)

Of course, ecthesis does mean this; this the second of the three meanings of ἔκθεσις mentioned above, and Smith simply denies that it can be used in the other senses. But if "proof by ecthesis" means no more than "proof by symbolization," why does Aristotle mention it so infrequently? Why is not every proof a proof by ecthesis?

To explain this, Smith backs away from the strongest version of his argument, saying instead, "The 'proofs through ecthesis' are so called because [in them] use of letters is involved in a special way" ("Ecthesis" 113).

Smith's position seems too weak to explain passages such as, "It is necessary for P to belong to some R (for a deduction through the first figure comes about). It is also possible to carry out the demonstration through an impossibility or through ecthesis" (28a20–22; emphasis added). If ecthesis is symbolization (and hence necessary for every proof), why is it presented as an option to a demonstration already given using symbols?

Smith may have seen the difficulties with his view, since in his more recent works he endorses Łukasiewicz's position, while accepting that it has limitations. He presents Łukasiewicz's ecthetic inference rules (L1–L4) without directly attributing them to Łukasiewicz and accepts the view as his own: "The interpretation I offer of ekthetic proof is not without its problems" ("Introduction" *xxiv*).

Ecthesis and Quantificational Logic

Treatments of Aristotle's system now universally discuss his logic in terms of modern quantificational logic. Łukasiewicz is blunt in his opinion on the necessity of studying Aristotle through a quantificational lens:

Prantl and Maier are now dead, but perhaps it would not be impossible to persuade living philosophers that they should cease to write about logic or its history before having acquired a solid knowledge of what is called "mathematical logic." It would otherwise be a waste of time for them as well as for their readers. (47)

Łukasiewicz's counsel is now universally followed. For example, Smiley hints that Aristotle must have known something of modern quantificational logic when he explains ecthetic proof through existential instantiation. Łukasiewicz and Patzig are unable to explain ecthesis without quantificational logic, but they deny that ecthesis refers to particular terms as required by existential instantiation (Łukasiewicz 61; Patzig 161). Even Smith insists that Aristotle knew universal generalization and, though his theory makes no direct use of it, existential instantiation as well.[11]

The standard positions can be seen as insistence on existential quantification on the one hand, and insistence on universals on the other. But, as has been shown, neither of these extreme views can explain all the instances of ecthetic proof in the text.

A fuller version of quantificational logic, however, explains well all passages in which Aristotle mentions ecthesis. The quantificational proofs for *Bocardo* and *Darapti* are given above. If ἔκθεσις is assumed to refer to quantificational logic in general,[12] the difficulties of Smiley's interpretation are avoided. Likewise, it can be shown that in its full strength quantificational logic can prove any valid syllogism.

Smith also criticizes Smiley's interpretation for not involving the normal syllogistic inference rules: "Aristotle says expressly in [chapter] 8 that 'each of the syllogisms comes about in its own figure' (30a13–14). This is easily explained in Łukasiewicz's interpretation.... However, if we explain ecthesis via ∃-elimination, syllogistic moods are not involved at all" ("Ecthesis" 118). He adds, "Patzig...makes much of this point, treating the argument in [chapter] 8 as a confirmation of Łukasiewicz's interpretation analogous to the discovery of a hitherto unknown manuscript" (118n).

---

[11]Smith's views have apparently changed since "Ecthesis," but I think he would regard the thesis of this essay as supporting that. However, he seems to try to distinguish the logical and metalogical elements in Aristotle's syllogistic but is himself confused by the distinction. He gives, therefore, conflicting accounts of the meaning of ecthesis, and his example of ecthetic proof does not conform to any of his explanations.

[12]As discussed above, we must also understand $\forall x(Ax \to Bx)$ to imply $\exists x Ax$ in order to avoid the problems with Smiley's interpretation and make *Darapti* and *Felapton* valid arguments.

If Aristotle considered ecthesis to be an alternative form of proof to his syllogistic moods, as quantification is a (far superior) alternative to syllogism, ecthetic proofs surely would not involve the normal syllogistic inferences.

One difficulty that arises with most interpretations of ecthetic proof is that Aristotle does not mention it (though he mentions proofs by reduction and impossibility) when he summarizes his system in chapter 7. However, assuming that Aristotle understood some of the differences between quantificational and syllogistic logic, one wouldn't expect him to explain it in a summary of syllogism. Also, if, as Łukasiewicz argues, Aristotle believed that syllogism (as opposed, I argue, to quantificational logic) had ontological implications (Łukasiewicz 5–7; cf. Smith, "Introduction" *xiv*), one would expect him to neglect the study of quantificational logic. Indeed, if "the *Prior Analytics* [is] a theoretical preliminary to the *Posterior*" (Smith, "Introduction" *xiii*), anything not directly related to this goal would not receive much attention.

Łukasiewicz says:

> There are many...points in the *Prior Analytics* suggesting that the contents of this work grew during its composition. Aristotle did not have time to draw up systematically all the new discoveries he had made, and left the continuation of his logical work to his pupil Theophrastus. (27)

While Łukasiewicz is here discussing Aristotle's knowledge of the fourth figure, his explanation accounts just as well for the minor role of ecthesis in Aristotelian theory. That is, Aristotle understood about quantization, but apparently didn't understand that its flexibility allows a system of logic far more powerful than syllogistic logic.

In conclusion, most of the theories of Aristotle's ecthetic proof assume a limited knowledge of quantificational logic, but they all fail to explain ecthesis either logically or textually. It seems certain that Aristotle did not have a fully developed theory of quantificational logic, nor did he understand its flexibility and power. His uses of ecthesis can best be explained, however, by assuming that Aristotle understood a great deal about quantization (both instantiation and generalization, both existential and universal) but that he left the system as a whole undeveloped.

Works Cited

Aristotle. *Analytica Priora*. Trans. A. J. Jenkinson. *The Basic Works of Aristotle*. Ed. Richard McKeon. New York: Random House, 1941. 62–107.

———. *Prior Analytics*. Trans. Robin Smith. Indianapolis: Hackett University Press, 1989.

Church, Alonzo. *An Introduction to Mathematical Logic, Vol. 1*. Princeton: Princeton University Press, 1956.

Clark, Michael. *The Place of Syllogistic in Logical Theory*. Nottingham: University of Nottingham Press, 1980.

Lear, Jonathan. *Aristotle and Logical Theory*. New York: Cambridge University Press, 1980.

Liddell, Henry George, and Robert Scott. *A Greek-English Lexicon*. New York: Oxford University Press, 1996.

Łukasiewicz, Jan. *Aristotle's Syllogistic from the Standpoint of Modern Formal Logic*. 2nd ed. Oxford: Clarendon Press, 1958.

Patzig, Günther. *Aristotle's Theory of the Syllogism: A Logico-Philological Study of Book A of the* Prior Analytics. Trans. Jonathan Barnes. Dordecht, Holland: D. Reidel Publishing, 1968.

Smiley, Timothy J. "Syllogism and Quantification." *Journal of Symbolic Logic* 27 (1962): 58–72.

Smith, Robin. Introduction. *Prior Analytics vii–xxxi*.

———. "What Is Aristotelian Ecthesis?" *History and Philosophy of Logic* 3 (1982): 113–27.