

## Learning to Count Again: On Arithmetical Knowledge in Kant's *Prolegomena*

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*One might indeed think at first that the proposition  $7+5=12$  is a merely analytic proposition, which follows according to the principle of contradiction from the concept of a sum of seven and five. But if we look more closely, we find that the concept of the sum of 7 and 5 contains nothing further than the unification of the two numbers into a single number, and in this we do not in the least think what this single number may be which combines the two. The concept of twelve is in no way already thought by merely thinking this unification of seven and five, and though I analyse my concept of such a possible sum as long as I please, I shall never find the twelve in it.*

Kant (PFM §2, 19)

**K**ant's *Critique of Pure Reason*<sup>1</sup> (*Critique*) was published in 1781. For the next few years, it was met with relative silence, and where not with silence, by non-committal responses and indications that the text was obscure, unintelligible, and largely unreadable (PFM 10). To rectify this state of affairs, and to make his project more accessible, Kant

<sup>1</sup>When citing the *Critique of Pure Reason* (CPR), I will follow the tradition of referring to the first edition of the *Critique* as A and the second edition as B, and I will follow Norman Kemp Smith's translation (1965).

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published the *Prolegomena to any Future Metaphysics*<sup>2</sup> (*Prolegomena*), in 1783. The *Prolegomena*, however, was not a mere summary of the *Critique*, but was also a new approach toward the topic: an approach which Kant considered “analytic,” in contrast to the “synthetic method” of the *Critique*. Evidently, Kant considered this new approach a success, as great portions of the *Prolegomena* are to be found in his second edition of the *Critique*, published in 1787.

This paper concerns Kant’s account of mathematical knowledge in his *Prolegomena*. Specifically, it concerns whether mathematical knowledge is both a priori and synthetic. In his *Critique*, Kant praises mathematics as “the most splendid example of the successful extension of pure reason,” and values it as a key example of the synthetic a priori (CPR A712–B740). However, in the *Prolegomena*, Kant’s examination of mathematics plays a more important role: mathematics is that from which he compounds pure natural science and the universal laws of nature (PFM §15, 53).<sup>3</sup> Hence, Kant’s epistemology in the *Prolegomena* relies heavily on the apriority and syntheticity of mathematical knowledge. In this paper, I argue that Kant’s account of mathematical knowledge lends more readily to analyticity than syntheticity. If this is the case, then Kant’s attempt to account for metaphysical knowledge is undermined and the general argument for synthetic a priori knowledge is hindered. In what follows, I explicate Kant’s arguments for synthetic a priori mathematical knowledge and demonstrate how they implicitly entail analyticity.

This paper has five sections. Section one briefly contextualizes Kant’s project. Section two outlines Kant’s arguments for the syntheticity of arithmetical knowledge. In section three, I argue that Kant’s account of synthetic arithmetical knowledge is undermined by his own account of numbers. In section four, I demonstrate how these conclusions hold for the *Critique*. Finally, I provide a brief summary and conclusion in section five.

<sup>2</sup>When citing the *Prolegomena to any Future Metaphysics* that will be able to Present Itself as a Science (PFM), I will refer by section and page numbers to the translation by P. G. Lucas (1953).

<sup>3</sup>It is worth noting that mathematical knowledge is treated in a similar fashion in the *Critique*. In nearly every respect, Kant’s treatment of mathematical knowledge itself is identical between the texts—as is to be expected, given the role that the *Prolegomena* plays in his discourse. However, the ways in which mathematical knowledge interacts with his system differs between them. In the *Prolegomena*, mathematical knowledge is given a more fundamental role, from which other concepts are derived, whereas it is used in the *Critique* more to situate pure reason and to provide an example against which other forms of knowledge might be compared. In this paper, I focus primarily on the *Prolegomena*, though §4 will provide brief remarks on the relation between the two.

## I. On Mathematical Knowledge in Kant's *Prolegomena*

Kant's intent in the *Prolegomena* is to locate the sources of metaphysical knowledge. The reason for examining knowledge concerning metaphysics, according to Kant, is Hume's scepticism (PFM 6–9). Hume concluded that there is no a priori connection between apparent causes and effects, because the necessity of a cause cannot be contained in the concept of the effect. From this, Hume is said to have inferred that there can be no a priori metaphysical knowledge at all. The apparent epistemological problems which fall out of Hume's conclusion rely on the assumption that a priori knowledge requires analyticity: that the knowledge which arises without experience must arise from the content of the concepts considered alone, and that no new knowledge can be generated therefrom.<sup>4</sup> If it is possible to determine the existence of a priori knowledge which is not analytic, then metaphysical knowledge may be deemed possible (PFM 9–10). This, then, is Kant's project.

Indeed, Kant believes that mathematical knowledge satisfies these very conditions. In his *Prolegomena*, Kant finds mathematical knowledge—divided into arithmetic and geometry—to be a priori and synthetic. Furthermore, Kant believes that the knowledge of the existence of things (“the science of nature”) depends on the application of mathematics to appearances, and that this knowledge is also synthetic and a priori (PFM §14, 53). From these two accounts of synthetic a priori knowledge, Kant infers the possibility of the existence of other synthetic a priori knowledge, and so the possibility of metaphysical knowledge (PFM §4, 28–30). However, in what follows, I will argue that Kant's account of mathematical knowledge lends more readily to analyticity than syntheticity.

## II. Kant's Account of Mathematical Knowledge

Kant's analysis of mathematical knowledge is split into analyses of geometric knowledge and of arithmetic knowledge. Today, it is widely held that Kant's account of synthetic a priori geometric knowledge is undermined

<sup>4</sup>For Kant, those propositions which are analytic are “explicative” in that they add nothing new to the content of knowledge, and those which are synthetic are “ampliative” in that they do provide new knowledge to the content. See PFM §2, 16. Between his works, analyticity and syntheticity are categorized in different ways, but his ampliative account is recurrent, and for the purposes of this paper we will consider these terms defined as such.

by the later acceptance and advancements of non-Euclidean geometry.<sup>5</sup> In light of this, very little attention has been given to his arithmetic knowledge; presumably it is taken for granted that the refutation of Kant's geometric knowledge sufficiently undermines his system. Yet, Kant couches both time and space in terms of arithmetic knowledge, and it is not clear that the apriority and syntheticity of geometric knowledge are required for Kant's system to function (PFM §§10–3, 38–51; CPR A162–66–B202–07).

Finally, it is worth noting that Kant limits the scope of his analysis to *pure* mathematics—those mathematics which are a priori by nature—and so seeks no proof of apriority (PFM §2, 18–9). Accordingly, this paper will focus solely on Kant's arguments for the syntheticity arithmetic knowledge. Kant provides two arguments for the syntheticity arithmetic knowledge: a negative account against analyticity of arithmetic knowledge, and a positive account in favor of synthetic arithmetic knowledge. The following two paragraphs offer these arguments respectively.

To elucidate Kant's negative argument, we will take up his example of  $7+5=12$  (see PFM §2, 19; CPR A164–B205). According to Kant, the concept of the number twelve is not contained in the concept of the sum of seven and five. Rather, all that is contained in the concept of the sum of seven and five is the concept of the union of those numbers: there is no actual number twelve contained within those two numbers. However, it is not necessarily clear that 12 does arise in an ampliative way from  $7+5$ . This can be made clearer, Kant says, by taking up larger values. Certainly, when we look at  $64,523+7,684,421$ , the sum of those numbers is not contained within it, and surely that sum, when determined, is in some way new to us. Accordingly, deriving their sum is not merely explicative, but ampliative.<sup>6</sup> Thereby, Kant reasons that arithmetic is synthetic and not analytic.

Kant's positive argument for synthetic a priori arithmetic relies on the fundamental claim that arithmetic procedures depend on representations of the numbers and other arithmetic constituents: "But we find that all mathematical knowledge . . . must first exhibit its concept in intuition" (PFM §7, 36). That is, when we perform arithmetic functions, we do so with a representation in our minds: the number five might be represented by a certain number of fingers held, or the beads of an abacus. However, in order to align with the project at hand, such representations must be available a priori. For, if the representations rely on empirical experience,

<sup>5</sup>See, for example, Jones (1946), Wiredu (1970).

<sup>6</sup>See footnote four.

then the arithmetic cannot be itself a priori. Accordingly, such representations cannot be representations of objects. Hence, the matter is to determine whether there are any a priori intuitions or representations that can be used in arithmetic functions, which precede any experience of objects which they might themselves represent.

The answer, according to Kant, is to be found in the a priori intuitions of space and time. On his account, we require no a posteriori concepts or experiences to fuel our intuitions of space or time: they are the intuitions which remain when everything empirical is stripped from our perceptions (PFM §10, 39). In particular regard is time, for Kant believes that the concepts of numbers are formed from the successive addition of units in time (PFM §9, 39). That is, the concept of two arises from the successive intuition of two units in time. Accordingly, there are arithmetic functions which can be considered synthetic a priori, in that they do not rely on experience and are ampliative in nature.

Kant's conclusion about arithmetic and mathematics is novel, as it distances itself from a trend toward treating mathematics as analytic a priori. In particular, Kant believes that Hume held this latter position, and so mathematics offers a retort against the Humean scepticism (PFM §2, 21).<sup>7</sup> However, to note that Kant provides a novel treatment of arithmetical knowledge is not necessarily to say that he provides a convincing account.

### III. Against Kant's Account of Synthetcity

Kant provides a conception of arithmetic in which numbers are formed from the successive addition of units in time (PFM §10, 39; see also CPR A145–B185). That is, the number five is represented by a mental process which adds units in the same way that we might count fingers. It is reasonable, then, to assume that the function of addition uses the same temporal processes as the synthesis of numbers. Indeed, this seems particularly appropriate, given Kant's definition of numbers by appeal to addition. So, the process of adding seven and five involves something like counting out seven units and then counting out a further five. In doing so, we come to the conclusion that seven and five is equal to twelve, and Kant is inclined to say that this conclusion—that their sum is twelve—is something ampliative: it adds something more to our knowledge than was contained in  $7+5$  alone. However, it is not immediately clear how this conclusion

<sup>7</sup>Humean scholars, however, agree that there is room to interpret Hume as providing a synthetic a priori account of mathematics. See, for example, Steiner (1987).

could be ampliative, if the concepts of seven, five, and of addition, are attended to with apriority. In what follows, I will raise objections to Kant's claims that arithmetic is synthetic or ampliative.

First, it is not immediately clear that Kant's negative account is convincing. Kant's response to the concern that  $7+5=12$  does not appear to be ampliative is to suggest that we need to look at larger values: that the sum of  $64,523+7,684,421$  would be something of new knowledge to us, and that we find  $7+5=12$  to be evident only because (i) the numbers are small, and so any successive additions in time do not take long to perform, and (ii) we are familiar with the addition of those smaller values. Yet, it is not clear that surmising the sum of  $64,523+7,684,421$  is necessarily synthetic. For, if we were to establish the result ( $7,748,944$ ), it would not in any way seem to take us by surprise. Nor would we find ourselves inclined to say that we had learned something new. Yet, while it is worth raising, this is not the objection with which we will concern ourselves here.

For sake of clarity, let us symbolize the successive addition of units in time by successive squares ( $\square$ ), such that five is represented by  $\square\square\square\square\square$  and seven by  $\square\square\square\square\square\square$ . It seems that Kant's accounts of numbers is such that we constitute five by starting with  $\square$ , and successively adding further units to get  $\square\square$ ,  $\square\square\square$ ,  $\square\square\square\square$ , and finally  $\square\square\square\square\square$ . Note that five here is one unit more than four, or "four plus one." This seems trivial at first glance. However, if a process of successive addition gives us a priori access to the concepts of numbers such as four and five, then it is difficult to maintain that we generate any new knowledge from adding four and one to yield five. For if we recognize that four is  $\square\square\square\square$  and five is  $\square\square\square\square\square$ , and if we acknowledge they differ only by  $\square$ , then it should come as no surprise if it is not new knowledge that  $\square\square\square\square$  and  $\square$  together are  $\square\square\square\square\square$ . The same surely holds for larger numbers as well. In fact, we can apply the same reasoning to any additive scheme by recursive addition. For example, if we recognize that two is one plus one, then we can add two to any number by adding one twice. And, indeed, this meets our common intuitions about learning addition: children are frequently taught addition by counting conglomerates of unit cubes. Here, it seems, Kant has merely done away with the cubes in favour of an a priori succession in time, but has continued with the counting. If this is what constitutes addition, then it appears no different than merely counting or representing numbers, and it remains unclear how it can provide for any new knowledge. In this way, it seems that Kant's arithmetic is more analytic than it is synthetic.

One might object that this successive or recursive addition provides no information about the resultant sum.  $7+5$  contains only the concepts of seven, five, and addition, as well as implying the resultant union—but not the resultant sum. At most, it says that you count out seven successive units

in time, and then five more, but not what that process results in. There seems to be a problem analogous to the infamous problem of the unity of the proposition: how can the concepts there contained also contain the concept of their sum? However, this seems to say that knowing 12 is different from knowing  $\square\square\square\square\square\square\square\square\square$ : that knowing a number is not the same as knowing it as resultant recursive addition. Yet, Kant seems to treat the concept of twelve just as that concept of succession we mark by  $\square\square\square\square\square\square\square\square$ . The interlocutor may respond by claiming that the concept of twelve is a distinct concept which must be learned as  $\square\square\square\square\square\square\square\square$ , such that the concept is not contained by the recursive addition. Yet, this response fails in that (i) it rejects Kant's explicit definitional account,<sup>8</sup> and (ii) it requires that no number is known until it is learned as that succession, which would require an appeal to experience—to that *learning*—such that arithmetic cannot be treated as a priori, as the Kantian requires. If the definitional aspect is taken as irrelevant, and 12 just is  $\square\square\square\square\square\square\square\square$ , then the same problem arises: it seems that no new knowledge arises from the generation of  $\square\square\square\square\square\square\square\square$  as the sum of  $\square\square\square\square$  and  $\square\square\square$ , as distinct from the generation of  $\square\square\square\square\square\square\square\square$  alone.

However, a reader of the *Critique* might find our emphasis on the successive account to be unnecessary, for Kant does say that arithmetic “abstracts completely from the properties of the object that is to be thought in terms of such a concept of magnitude. It then chooses a certain notation for all constructions of magnitude as such (numbers), that is, for addition, subtraction, extraction of roots, etc.” (CPR A717–B745).<sup>9</sup> In reading this, one might be inclined to think that the successive addition account is not necessary for arithmetic, but that we may “choose a certain notation”

<sup>8</sup> Furthermore, there is no room here for Kantians to appeal to definitions being different from the concepts they explain, for Kant makes very explicit what he means by definition in both editions of the *Critique* (suggesting no change between): “To define, as the word itself indicates, really only means to present the complete, original concept of a thing within the limits of its concept” (CPR A727–B755). “Consequently, mathematics is the only science that has definitions. For the object which it thinks it exhibits a priori in intuition, and this object certainly cannot contain more or less than the concept, since it is through the definition that the concept of the object is given” (CPR A729–B757).

<sup>9</sup> It is worth noting for the non-mathematician, that processes such as multiplication and division can be defined by means of addition and subtraction, and hence addition alone. To divide one number by another is to see how many successive additions of the latter number it takes to reach the former. Indeed, computational algorithms are often reduced to additive and subtractive schemes. That there may be shortcuts is just to learn the patterns and rules associated with these means, but it should be clear that successive addition is capable of accounting for different arithmetical functions, and so is not restricted to our  $7+5=12$  example alone.

which precludes the difficulties raised above. But, this is to put the cart before the horse: Kant repeatedly indicates—in both the *Prolegomena* and the *Critique*—that numbers just are those successive additions through time (PFM §10, 39; CPR A45-9–B62-6, A163-66–B204-07, and A724–B753). Indeed, it seems crucial to his very account of the pure intuition of time to do so. To take this quotation in isolation from Kant's insistence of the successive addition account is to misrepresent Kant's mathematical account on a whole, for it is by means of successive addition that he is able to appeal to the apriority of mathematics.

Finally, it is worth responding to the very claim which Kant levies against this attack. In the *Critique*, but not explicitly in the *Prolegomena*, Kant defends his positive account of syntheticity from objections of the kind raised here. In claiming that  $7+5=12$  is synthetic, he notes that the concept of twelve is not included in the concepts of seven, of five, or of addition. "That I must [find the concept of twelve] in the *addition* of the two numbers is not the point, since in the analytic proposition the question is only whether I actually think the predicate in the representation of the subject" (CPR A164–B205).<sup>10</sup> It seems that Kant is treating addition as a function which acts upon five and seven, to produce twelve, and that the addition sign in  $7+5$  merely represents this function. Hence, if the proposition represents only the function of addition, then the seven and the five do not come together to make twelve. But, there seem to be a few problems here.

Principally, the concepts of seven and five are necessarily attended to with successive addition, as per the definition which Kant sets out, such that we can conceive of each of them as the output of addition functions:  $7+5$  is just  $(1+1+1+1+1+1+1)+(1+1+1+1+1)$ . Here, the law of association finds this precisely equal to the definition of twelve. Accordingly, in order to have the pure concepts, there must be successive addition, and hence, successive addition is present in the proposition if there are numbers. It remains unclear what categorical difference exists between the successive addition of units in time which defines a number and the successive addition of units in time which defines a sum. In this way, it is difficult

<sup>10</sup>Few authors have approached Kant's arithmetical knowledge, presumably given that (i) the geometric knowledge was amply refuted by non-Euclidean models, and (ii) we do not seem to psychologically do mathematics in this way. Those who have studied Kant's arithmetic, however, have tended to focus their attention on the Introduction to the *Critique* and to "Transcendental Doctrine of Method" (see Brittan (2006)). Accordingly, where there are discussions of Kant's arithmetic, they tend to miss Kant's defense.

to see why 12 is not present in the proposition. Indeed, Kant's recursive treatment of numbers seems to permit exactly that.

An appeal to common intuition finds the same result. If we are presented with  $7+5$ , and if the definitional account of numbers Kant employs is accurate, then we would read or engage with the proposition by means of successively adding units of time for 7 and then successively adding units of time for 5. By whatever capacity we have which distinguishes those units of time into numbers, that period should bring to mind 12. Let us return to the notion of children learning arithmetic with unit cubes. A child is presented with the task of preparing for  $7+5$  (preparing in the sense that we are not yet adding). The child sets out seven unit cubes, one by one, followed by five more. In setting out these cubes, the child has set out twelve cubes, and the duration which it took to count out the cubes for both numbers is equal to the duration it takes to count out their sum, *ceteris paribus*. Now, if a concept of a number is wholly explained by this successive addition, as Kant thinks, then the child, in setting out  $5+7$  alone has also set out 12, and so the arithmetical proposition is not ampliative and thus not synthetic. In this way, it is not clear how Kant expects to hold the distinction which rebuts the objections raised against his account. It may be that a Kantian mathematician can work out the distinctions between the kinds of addition herein, but at present, Kant's own defence against our objections raised seems at odds with his other proposals.

Kant's own successive addition account of numbers critically undermines his arguments for the syntheticity of arithmetical knowledge. At this point, we may see fit to conclude that his larger project fails. For, where Kant derives metaphysical knowledge from the fact that mathematical knowledge is synthetic a priori, it has been shown that Kant's mathematical knowledge is not in fact synthetic a priori. However, that this conclusion holds for his *Prolegomena* does not mean that it holds also for the *Critique*: the *Prolegomena* is only a prolegomena. In what remains, I will demonstrate that these conclusions hold also for Kant's *Critique*.

#### IV. Consequences for the *Critique*

The discussion in this paper has focused on arithmetic in Kant's *Prolegomena*, wherein arithmetic plays a critical foundational role in the context of Kant's greater project. In the *Critique*, mathematical knowledge plays a merely exemplary role, rather than a foundational one (see CPR A48-9-B66, A707-38-B735-66). Here, in the *Critique*, he takes mathematical knowledge to be the poster boy for the "successful extension of pure reason" (CPR A712-B740). In this manner, there may be a concern

that the problems here raised do not carry from the *Prolegomena* to the *Critique*. In what follows, I will briefly indicate why they do carry.

Kant's concepts of mathematics and of number do not change between editions, nor do they differ from the *Prolegomena*. Kant relates arithmetic to space and time in the *Critique* in a similar fashion as in the *Prolegomena* (see CPR A45-9-B62-6, and A163-66-B204-07), as well as his account of numbers as successive addition of units in time (see CPR A45-9-B62-6, and A724-B753). In this way, with the same system in place for the analysis of mathematical knowledge in both the *Prolegomena* and the *Critique*, it seems that objections against mathematical knowledge in one text are also objections to the other. The question, then, is to what extent these objections undermine Kant's project in the *Critique*. For, while we have focused on the *Prolegomena* for purpose of clarity in attacking Kant's system, it would do little to attack only his prolegomena in this capacity.

Indeed, there is cause to concern the *Critique* with these objections. For, while mathematical knowledge is not used critically in the exploration of the pure concepts of space and time, it is referred to with a particular priority. Here, it is worth quoting at length from the "Transcendental Doctrine of the Method":

Mathematics presents the most splendid example of the successful extension of pure reason, without the help of experience.... Thus pure reason hopes to be able to extend its domain as successfully and securely in its transcendental as in its mathematical employment, especially when it resorts to the same method as has been of such obvious utility in mathematics. It is therefore highly important for us to know whether the method of attaining apodeictic certainty which is called *mathematical* is identical with the method by which we endeavour to obtain the same certainty in philosophy, and which in that field would have to be called *dogmatic*. (CPR A712-13-B740-41)

It is clear, then, that Kant places significant value on mathematics in his system in the *Critique*: it is a standard against which other measures of pure reason might be compared. This, coupled with mathematical knowledge's clear relevance and employment in the determination of the pure intuitions of space and of time, mark mathematical knowledge as of great import in the *Critique* (see CPR A45-9-B62-6, and A163-66-B204-07). These points in mind, it is easy to conclude that the attacks raised here stand also against the Kantian corpus, and not the *Prolegomena* alone. However, the specific consequences for the *Critique* require a deeper treatment of the *Critique* itself, and are beyond the present scope of this paper.

## V. Summary and Conclusion

In this paper, I explored Kant's mathematical knowledge in his *Prolegomena*. In particular, I demonstrated that his account of the syntheticity of arithmetic stands in direct contention with his account of numbers. For, it was found that both arithmetical processes and numbers can be couched in terms of the successive addition of units in time, such that a sum (or other resultant) is already determined by the arithmetical proposition. This position was then defended against objections, and shown to carry some import against Kant's account in the *Critique*.

The arguments presented here, when coupled with the arguments from non-Euclidean geometry against Kant's geometric knowledge, undermine Kant's notion of mathematics as a priori and synthetic. And, as the *Prolegomena* relies heavily on mathematics, Kant's project in the *Prolegomena* seems to falter on a whole. Granted, it may be that a thorough analysis of Kant's earlier works on mathematics—accounts which were somewhat more detailed, but from which he moved away in the *Critique*—could provide a way out for Kant, or that a reworking of the Kantian categories and notions of pure intuition may put to rest the concerns raised here. However, that burden falls to the Kantian to defend, for there does not appear to be any support for such a case in the *Prolegomena* or the *Critique*. Thus, the strongest contender for synthetic a priori knowledge has fallen out of the running, and Kant's whole project has been jeopardized by his own account of mathematics.

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